THE MATHEMATICAL THEORY OF INVESTMENT

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BY

ERNEST BROWN SKINNER

ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF WISCONSIN

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PREFACE

This book has been written for the use of students of business and public affairs. The fact that, with a few notable exceptions, colleges and universities have hitherto made no provision for courses in mathematics adapted to meet the needs of students trained for commercial careers and for the public service, is due in no small measure to the lack of a suitable textbook. In recent years the closer study of business methods and the extension of governmental control over many forms of industrial and financial activity have greatly emphasized the value, to students of finance, of a knowledge of what the Germans call "political arithmetic."

In an attempt to meet the need for a book covering this field, some of the more important topics relating to the theory of interest and its application to the larger affairs of modern everyday life have been brought together. The following pages contain, in somewhat elaborated form, the substance of a course of lectures given annually for the last five years to the students in the course in commerce in the University of Wisconsin. While I have tried to make the book practical throughout, I have sought to make it a book of first principles rather than a guide to detailed practice. I trust that the reader will find the treatment adapted to present-day conditions and needs. I shall be gratified if I have in any way contributed to the more efficient training, in principles of sound finance, of the splendid body of young men who are going out from our courses in commerce to become leaders in the business world.

In selecting topics for consideration, an effort has been made to avoid those that are still the subject of controversy. If this rule has been departed from in devoting some space to depreciation, it is on account of the great importance of the subject. The public-utility commission seeking to make an equitable adjustment of rates must make some sort of quantitative determination of depreciation, even though neither the exact meaning

of the term nor the method of treating the subject has been agreed upon. It is believed that the discussion in §§ 56–58 follows logically from the definitions therein proposed. It might be added that it is in substantial agreement with the practice of the Wisconsin Railroad Commission.

Much of the inspiration for the book has come from M. Cantor's admirable "Politische Arithmetik," while most of the material may be found in some form in Todhunter and King's "Institute of Actuaries' Text-Book." I have consulted freely Fuzet and Reclus, "Précis de mathématiques commerciales et financières"; Martini, "Aritmetica commerciale e politica"; Schlimbach, "Politische Arithmetik"; Wolff, "Inheritance Tax Computations"; Broggi, "Matematica attuariale"; Löwy, "Versicherungsmathematik"; Dawson, "Practical Lessons in Actuarial Science"; and Willey, "Principles and Practice of Life Insurance," revised by Moir.

Tables III–VII are based upon Spitzer's "Tabellen für Zinses-Zinsen- und Rentenrechnung," and have been carefully compared with Vintéjoux's "Nouvelles tables d'intérêts composés et annuités." Tables XI and XII have been taken, with the kind permission of The Spectator Company, from the books of Dawson and Willey.

In preparing the manuscript, I have had the benefit of the valued criticism of my colleagues, Professors E. B. Van Vleck and L. W. Dowling, and of Mr. L. A. Anderson, Actuary for the Wisconsin Insurance Department. Professor Arnold Dresden, Dr. Florence Allen, Dr. G. R. Clements, and Mr. T. M. Simpson, who have tried out a large part of the material in the classroom, have made many helpful suggestions in the selection of material and examples, and have rendered valuable assistance in verifying worked-out examples and in correcting proofs. I am also under obligations to Professor D. F. Campbell, of Armour Institute, for placing at my disposal notes and other material relating to the elements of the theory of life insurance.

ERNEST B. SKINNER

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THE MATHEMATICAL THEORY OF INVESTMENT

PART I. ALGEBRAIC INTRODUCTION

CHAPTER I

PROGRESSIONS

1. Definitions. An *arithmetical progression* is a succession of terms such that any term may be obtained from the preceding term by the addition of a constant number called the *common difference.* If the common difference is positive, the progression is said to be *increasing;* if it is negative, the progression is said to be *decreasing.*

ILLUSTRATIVE EXAMPLES. The progressions

$$2,\quad 4,\quad 6,\quad 8$$

and

$$1,\quad 1\tfrac{1}{2},\quad 2,\quad 2\tfrac{1}{2}$$

are increasing arithmetical progressions, the first with common difference 2 and the second with common difference $\frac{1}{2}$.

The progression

$$8,\quad 6,\quad 4,\quad 2,\quad 0,\quad -2$$

is a decreasing arithmetical progression with common difference -2.

A *geometrical progression* is a succession of terms such that any term may be obtained by multiplying the preceding term by a constant number. The constant multiplier by means of which any term is derived from the preceding term is called the *ratio.* If the ratio is numerically greater than 1, the progression is called an *increasing* geometrical progression; if it is numerically less than 1, the progression is called a *decreasing* geometrical progression.

ILLUSTRATIVE EXAMPLES. The progression

$$1, \quad 2, \quad 4, \quad 8, \quad 16$$

is an increasing geometrical progression with ratio 2, while the progression

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{4}, \quad \tfrac{1}{8}, \quad \tfrac{1}{16}$$

is a decreasing geometrical progression with ratio $\frac{1}{2}$.

When a progression is given, the student should be able to determine quickly whether it is arithmetical or geometrical or whether it belongs to some other class. The test is supplied by the respective definitions.

A progression cannot be at the same time arithmetical and geometrical, except in the trivial case where the common difference is zero or the ratio is 1. For example, the succession

$$2, \quad 2, \quad 2, \quad 2$$

may be considered as an arithmetical progression with common difference zero or as a geometrical progression with ratio 1 (see Examples 12 and 13, § 2).

The important things to be considered in connection with progressions are the first term, the last term, the number of terms, the common difference if the progression be arithmetical or the ratio if it be geometrical, and the sum of all the terms. It will soon appear that any one of these five numbers may be expressed in terms of three others. In other words, if three of the five numbers are given, it is possible to find out all about the progression, provided it is known whether it is arithmetical or geometrical.

Progressions play a very important part in the theory of investment.

2. Derivation and use of formulas. The letters a, l, n, d, and s are used to denote the first term, the last term, the number of terms, the common difference, and the sum of all the terms of an arithmetical progression. The terms are then

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \cdots, \quad a+(n-1)d,$$

and the last, or nth, term is clearly given by the formula

$$l = a+(n-1)d. \qquad (A)$$

The sum of all the terms is

$$s = a + a + d + a + 2d + \cdots + a + (n-1)d,$$

or, if the terms be written in reverse order,

$$s = l + l - d + l - 2d + \cdots + l - (n-1)d.$$

If these two identities be added, the result will be

$$2s = (a + l) + (a + l) + \cdots \text{ to } n \text{ terms};$$

whence
$$s = \frac{n}{2}(a + l). \tag{B}$$

Similarly, the first term, the last term, the number of terms, the ratio, and the sum of all the terms of a geometrical progression are denoted by the letters

$$a, \quad l, \quad n, \quad r, \quad \text{and} \quad s.$$

The terms of the series are then

$$a, \quad ar, \quad ar^2, \quad \cdots, \quad ar^{n-1},$$

and the formula for the last, or nth, term is

$$l = ar^{n-1}. \tag{C}$$

The sum of n terms written out at length is

$$s = a + ar + ar^2 + \cdots + ar^{n-1}.$$

To obtain a formula for computing the sum, this equation is first multiplied by r. The resulting equation is

$$rs = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

If now the equation for s be subtracted from the equation giving the value of rs, the result is

$$rs - s = ar^n - a,$$

and the value of s obtained from this equation is

$$s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r}. \tag{D}$$

The expression ar^n is equivalent to $ar^{n-1} \cdot r$, and by equation (C)

$$ar^{n-1} \cdot r = rl.$$

If ar^n be replaced by its value rl, equation (D) takes the convenient form

$$s = \frac{rl - a}{r - 1}. \tag{D'}$$

By means of formulas (A) and (B), considered as a pair of simultaneous equations, any possible problem in arithmetical progressions may be solved when three of the numbers

$$a, \quad l, \quad n, \quad d, \quad \text{and} \quad s$$

are given. The resulting equations, when numbers are substituted for the three known quantities, will constitute either a linear system or a linear quadratic system, and a solution can be found by elementary algebra in every case.

In a similar fashion, problems in geometrical progressions may be solved by means of equations (C) and (D) or (D'). In certain cases, however, the simultaneous system cannot be solved by elementary means. This is always true when n is unknown, and usually true when r is unknown. Methods for obtaining a solution when n is unknown will be given in § 15.

The *arithmetical mean* of two numbers is defined to be one half their sum. It is easy to see that if three numbers are in arithmetical progression, the second is the arithmetical mean of the first and the third, for if a, b, and c are in arithmetical progression,

$$b = a + d \quad \text{and} \quad c = a + 2d;$$

so that
$$\frac{a + c}{2} = \frac{a + a + 2d}{2} = \frac{2(a + d)}{2} = b.$$

For example, $6\frac{1}{2}$ is the arithmetical mean of 5 and 8 and is the middle term of the arithmetical progression 5, $6\frac{1}{2}$, 8.

The *geometrical mean* of two numbers is the square root of their product. If three numbers are in geometrical progression, the second is the geometrical mean of the first and the third, for if a, b, and c are in geometrical progression, then

$$b = ar \quad \text{and} \quad c = ar^2.$$

Consequently,
$$\sqrt{ac} = \sqrt{a \cdot ar^2} = ar = b.$$

As an example, 6 is the geometrical mean of the numbers 3 and 12. Likewise $\sqrt{15}$ is the geometrical mean of the numbers 3 and 5.

In the solution of the examples the student should pay particular attention to the setting up of the pair of simultaneous equations from which the solution is obtained. It is advisable for the beginner to write down both equations, even though only one is needed.

EXAMPLES

1. The sum of a number of terms of a progression whose first three terms are 32, 24, and 16 is 80. How many terms has the progression?

Solution. The progression is arithmetical with the common difference $d = -8$. Furthermore, $a = 32$, $s = 80$. If these values for a, d, and s are inserted in equations (A) and (B), the equations become

$$l = 32 - (n-1)\,8,$$

$$80 = \frac{n}{2}(32 + l).$$

Eliminating l, which is not called for, we find the quadratic equation

$$n^2 - 9n + 20 = 0,$$

whose solution is given by $n = 4$ and $n = 5$. It is easy to see that the sum of four terms is the same as the sum of five terms, since the fifth term is 0.

2. The sum of eight terms of the progression of which three terms are 8, 16, 32 is 510. What is the first term?

Solution. In this problem the series is geometrical and $n = 8$, $r = 2$, $s = 510$. These values, substituted in equatious (C) and (D), give

$$l = a\,2^7$$

$$510 = \frac{a\,2^8 - a}{2 - 1}.$$

In this case the value of the required number a may be found from the second equation alone, but if the equations (C) and (D') had been used, it would have been necessary to determine a by means of the two equations

$$l = a\,2^7,$$

$$s = \frac{2\,l - a}{2 - 1}.$$

The result is the same whichever way the solution is found: viz. $a = 2$.

3. The first and last terms of an arithmetical progression are 4 and 100 respectively, and the difference is 4. Find the number of terms and the sum of all the terms.

4. The first three terms of a progression are 3, 6, and 9. Find the number of terms when the sum of all the terms is 234.

5. Suppose that in Example 4 the sum were 225 and the remaining data the same. What would be the value of n? What is the meaning of the result?

6. The first three terms of a progression are 3, 6, 12. Find the sum of 10 terms.

7. An elastic ball is dropped from a height of 20 feet, and each time it strikes the ground it rebounds to one half the distance through which it fell. How far will it have traveled when it strikes the ground for the tenth time?

8. Find the sum of the multiples of 3 that lie between 200 and 400.

9. A body falling from rest falls approximately 16.1 feet the first second, 48.3 feet the second, 80.5 feet the third, and so on. How far will it fall in 10 seconds?

10. The value of the timber in a certain forest increases at the rate of 4% annually. If it was worth $10,000 at the beginning of a five-year period, how much will it be worth at the end of the period?

11. Find the sum of five terms of the series

$$\frac{1}{\sqrt{3}+\sqrt{2}},\quad 5-2\sqrt{6},\quad 9\sqrt{3}-11\sqrt{2},\quad \cdots.$$

12. Prove that three numbers in arithmetical progression cannot be in geometrical progression unless the common difference is zero.

13. Prove that three numbers in geometrical progression cannot be in arithmetical progression at the same time unless the ratio is $+1$.

3. Geometrical progressions in which the number of terms increases indefinitely. In every problem in progressions so far considered the number of terms has been finite. There are, however, many important problems whose solution requires the concept of a decreasing geometrical progression in which the number of terms is indefinitely increased. An example will help to make the concept clear. Suppose a stick 2 feet long is cut into two equal parts, and one of these is cut into two equal parts, and so on, one of the two parts into which a given piece is cut being bisected each time. In this way a series of shorter sticks is obtained whose lengths are 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and so on, each fraction having for its

denominator a power of 2. These terms, as far as they go, form a geometrical progression whose ratio is $\frac{1}{2}$ and whose nth term is given by

$$l = 1 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}.$$

The sum of n terms will be

$$s = \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} = 2 - \frac{1}{2^{n-1}}.$$

Consider now what happens if the process of bisection is carried on indefinitely. Two things are clear from the nature of the problem: First, the length of the last piece becomes indefinitely small, or, to use a customary phrase, the length of the last piece *tends toward zero as a limit.* In the second place, however far the process is carried on, the sum of all the pieces cut off can never exceed 2, though it can be made to approach as near to 2 as we please. This is seen from the expression just found for s, viz.

$$s = 2 - \frac{1}{2^{n-1}}.$$

This relation may be written

$$2 - s = \frac{1}{2^{n-1}},$$

from which form it can be seen that the right member becomes small at will when n is increased, and consequently the difference $2 - s$ may be made as small as we please. In such a case it is customary to say that the limit of $2 - s$ is equal to 0, where the term *limit* is used in accordance with a definition to be formulated in the next chapter, and the relation is written

$$\lim_{n=\infty} (2 - s) = 0.$$

From this relation we deduce, by Theorem IV of § 4,

$$\lim_{n=\infty} s = 2,$$

and thus are brought back to the length of the stick in its original condition.

To generalize these notions for any geometrical progression whose ratio is numerically less than unity, consider the sum

$$s_n = a + ar + ar^2 + \cdots + ar^n,$$

where the subscript n is used to emphasize the fact that the sum of n terms is under consideration. By (D) of § 2,

$$s_n = \frac{a - ar^n}{1 - r},$$

which may be written $s_n = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$,

or $$\frac{a}{1-r} - s_n = \frac{ar^n}{1-r}.$$

If in this equation r be taken *numerically less than unity*, r^n will be smaller than r itself, and as n increases, r^n, and consequently $\dfrac{ar^n}{1-r}$, will tend toward zero as a limit. According to the notation used in the stick problem above,

$$\lim_{n=\infty}\left(\frac{a}{1-r} - s_n\right) = \lim \frac{ar^n}{1-r} = 0,$$

and finally $$\lim_{n=\infty} s_n = \frac{a}{1-r}. \qquad (E)$$

It is important to note carefully that equation (E) does not state that s_n is equal to $\dfrac{a}{1-r}$. The equation means just what it says, namely, $\dfrac{a}{1-r}$ is the limit toward which s_n is tending, and that by taking n large enough the difference between s_n and $\dfrac{a}{1-r}$ can be made as small as we please. Its importance lies in the fact that it furnishes a direct means for the solution of all problems like the stick problem. In that problem $a = 1$ and $r = \frac{1}{2}$. Substituting these values in equation (E),

$$\lim_{n=\infty} s_n = \frac{1}{1 - \frac{1}{2}} = 2.$$

Another example is the determination of the limit toward which a repeating decimal like $.333\cdots$ tends. The repeating decimal may be written in the form

$$.3+.03+.003+\cdots,$$

from which it is readily seen that the terms are in geometrical progression, with the first term $a=.3$ and the ratio $r=.1$. If these values be substituted in the right member of equation (E), the result is

$$\frac{.3}{1-.1}=\frac{1}{3},$$

which is, as we know, the common fraction that gives rise to the repeating decimal $.333\cdots$. In the same way we may find the common fraction which is the equivalent of any repeating decimal. For example, the repeating decimal $.23459459\cdots$ is equivalent to the decimal $.23$ increased by the geometrical progression

$$.00459+.00000459+\cdots,$$

whose ratio is $.001$; consequently,

$$.23459459\cdots=.23+\frac{.00459}{1-.001}=\frac{217}{925}.$$

A geometrical progression in which the ratio is numerically less than unity and in which the number of terms is increased indefinitely is called an *infinite geometrical progression.*

EXAMPLES

1. Find the limit of the sum for each of the following infinite geometrical progressions:

(a) $2+.5+.125+\cdots,$

(b) $\sqrt{2}+1+\frac{1}{\sqrt{2}}+\cdots,$

(c) $\sqrt{2}+1+1+\sqrt{2}-1+\cdots.$

2. What common fraction is the equivalent of the repeating decimal $.237237\cdots$?

3. Find the common fraction which is equivalent to the repeating decimal $.235959\cdots$.

4. A body is projected with an initial velocity of 25 miles per minute. If it goes $23\frac{3}{4}$ miles the second minute, $22\frac{9}{16}$ miles the third, and so on forever, how far will it go?

CHAPTER II

LIMITS AND SERIES

4. Variables and sequences. In mathematics it is necessary to consider two very distinct kinds of numbers, or quantities.

Numbers of the one kind do not change during the discussion of the problem in which they occur, and are called *constant numbers*, or simply *constants*. The numbers 2, 5, $\frac{1}{2}$, the first term of a progression, are constants. On the other hand, a number which takes different values in the discussion of a problem is called a *variable*. The number s_n of the previous section is a variable. Moreover, the number n itself is a variable upon which the value of s_n depends.

It is customary to use the first letters of the alphabet to denote constants, and the last letters to denote variables.

In the problems in which variables occur the values which the variables may take are usually restricted to *sets* of numbers following definite laws. Thus, in the infinite geometrical progressions the variable n may take the values 1, 2, 3, 4 $\cdots$. In the stick problem s_n took the values 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$ $\cdots$.

A set of numbers in which the successive numbers are determined according to some definite law is called a *sequence*. A variable which takes the successive values of a sequence is said to run through the sequence. The variable s_n in the stick problem runs through the sequence 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$ $\cdots$. Many operations of arithmetic lead us to sequences. For example, the attempt to reduce the common fraction $\frac{1}{3}$ to a decimal leads us to the sequence

$$.3, \quad .33, \quad .333, \quad \cdots,$$

whose terms form the successive approximations to the value of the fraction. Again, the process of root extraction will lead to the sequence

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad \cdots,$$

whose terms form the successive approximations to the value of $\sqrt{2}$. The extraction of any real root of any given number will lead to a definite sequence.

Another very important example of a sequence is the set of numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad \cdots,$$

which represent the successive stages in the attempt to find an approximate value of the ratio of the circumference of a circle to its diameter.

In all the examples of sequences which have just been given there exists a fixed number such that the difference between it and the numbers of the sequence becomes smaller and smaller. When a variable x runs through the numbers of such a sequence, the fixed number a is called the *limit of the variable*, in accordance with the following definition: *A fixed number a is called the limit of a variable x if, as x runs through the numbers of a sequence, the difference $a-x$ becomes and remains numerically smaller than any number that can be assigned.*

The relation between a variable x and its limit a is written

$$\lim x = a$$

and is read "the limit of x equals a."

The variable x may approach its limit in various ways. The values taken by x may be always less than a, or always greater than a, or sometimes less and sometimes greater. In the stick problem the values taken by s_n were always less than the limit 2; a variable which runs through the sequence

$$2\tfrac{1}{2}, \quad 2\tfrac{1}{4}, \quad 2\tfrac{1}{8}, \quad 2\tfrac{1}{16}, \quad \cdots$$

is always greater than its limit 2, while, if we consider the sum of n terms of the series

$$1 - .1 + .01 - .001 + .0001 - \cdots,$$

whose terms are the successive powers of .1 with signs alternately plus and minus, we obtain the sequence

$$1, \quad .9, \quad .91, \quad .909, \quad .9091, \quad .90909, \quad \cdots.$$

The first, third, fifth, and indeed all the odd-numbered terms of this sequence are greater than $\frac{10}{11}$, while the even-numbered terms are less than the same fraction, which is the limit of the variable that runs through the sequence.

Whatever may be the relation between a and x as x is approaching a, we write

$$|a-x|$$

to denote the numerical, as opposed to the algebraic, value of the difference between a and x. According to this notation $|3-5|$ and $|5-3|$ have the same value 2. A quantity inclosed between such bars is always positive.

In order that x shall approach the limit a, it is necessary and sufficient that $|a-x|$ should become and remain less than ϵ, however small ϵ may be chosen.

The following theorems, for which demonstrations may be found in books on algebra and analysis, are of great use in many branches of mathematics:

I. *If a variable x always increases but never takes values greater than a fixed number A, it approaches a limit which is either A or some number less than A. Similarly, if x always decreases but never takes values smaller than a fixed number A', it approaches a limit which is either A' or some number greater than A'.*

II. *If two variables are always equal and each approaches a limit, their limits are equal; i.e. if $x=y$, $\lim x=a$, and $\lim y=b$, then $a=b$.*

III. *The limit of the product of a constant and a variable is equal to the product of the constant and the limit of the variable. In symbols, $\lim cx = c \lim x$.*

IV. *The limit of the sum, difference, or product of two variables is equal to the sum, difference, or product of their limits. Or, $\lim(x+y)=\lim x+\lim y$, $\lim(x-y)=\lim x-\lim y$, $\lim(xy)=\lim x\cdot\lim y$.*

V. *The limit of the quotient of two variables is equal to the quotient of their limits, provided the limit of the divisor is different from zero, i.e. $\lim\frac{x}{y}=\frac{\lim x}{\lim y}$.*

VI. *The limit of the reciprocal of a variable whose limit is different from zero is equal to the reciprocal of the limit of the variable. In symbols,* $\lim \frac{1}{x} = \frac{1}{\lim x}$.

VII. *The limit of a power of a variable is equal to the power of the limit, provided we have not at the same time* $\lim x = 0$ *and* $n < 0$; *or,* $\lim x^n = (\lim x)^n$.

In solving problems for the determination of limits it is usually necessary to perform some algebraic reduction upon the expression whose limit is to be found, and then apply one or more of the foregoing theorems. For example, suppose it is required to find the limit of the expression $\frac{3x+5}{4x+2}$ as x becomes indefinitely large. If both numerator and denominator be divided through by x, we have the identity

$$\frac{3x+5}{4x+2} \equiv \frac{3+\frac{5}{x}}{4+\frac{2}{x}}.$$

Applying in succession Theorems II, V, IV, and VI, we find

$$\begin{aligned}
\lim \frac{3x+5}{4x+2} &= \lim \frac{3+\frac{5}{x}}{4+\frac{2}{x}} \\
&= \frac{\lim\left(3+\frac{5}{x}\right)}{\lim\left(4+\frac{2}{x}\right)} \\
&= \frac{3+\lim\frac{5}{x}}{4+\lim\frac{2}{x}} \\
&= \frac{3}{4}.
\end{aligned}$$

Again, suppose it be required to find the value of

$$\lim_{x=2}\frac{x^2-9x+14}{x^2-11x+18}.$$

The expression has no sense when $x=2$, for then it takes the form $\frac{0}{0}$, but both numerator and denominator of the fraction are divisible by $x-2$, so that

$$\lim_{x=2}\frac{x^2-9x+14}{x^2-11x+18}=\lim_{x=2}\frac{x-7}{x-9}=\frac{5}{7}.$$

EXAMPLES

1. Prove that $\lim\limits_{n=\infty} s_n = \lim\limits_{n=\infty}\dfrac{a-ar^n}{1-r}$

$$=\frac{a}{1-r}.$$

2. Find the value of $\lim\limits_{x=3}\dfrac{x^2-5x+6}{x-3}$.

3. Find the value of $\lim\limits_{h=0}\dfrac{[(x+h)^2-5(x+h)+6]-[x^2-5x+6]}{h}$.

4. Find the value of $\lim\limits_{h=0}\dfrac{(x+h)^n-x^n}{h}$.

5. Find the value of $\lim\limits_{x=\infty}\dfrac{3x^5+6x^4-7x^3+17}{11x^5+29x^3-43}$.

6. Find the value of $\lim\limits_{x=\infty}\dfrac{6x^7+3x^3-17x+20}{14x^6-7x^5+347}$.

7. Find the value of $\lim\limits_{x=\infty}\dfrac{29x^7-3x^6+14x^3}{31x^8+16}$.

8. What is the value of $\lim\limits_{x=\infty}\dfrac{a_0x^m+a_1x^{m-1}+\cdots+a^m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}$ in the following cases?

(a) $m>n$, (b) $m=n$, (c) $m<n$.

5. Infinite series with constant terms. Consider the infinite sequence of constant terms

$$u_1,\quad u_2,\quad u_3,\quad \cdots,\quad u_n,\quad \cdots.$$

The expression $\quad u_1+u_2+u_3+\cdots+u_n+\cdots$

is called an *infinite series.* Examples of infinite series have already been found in the section on infinite geometrical progressions. Other examples would be

$$(1)\ 1+2+3+4+\cdots.$$

$$(2)\ 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots.$$

$$(3)\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

$$(4)\ 1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.$$

In the case of the infinite geometrical progressions the series were of value because it was possible to find in each case a fixed number which was the limit of the sum toward which the sum of the first n terms of the series was approaching as n was increased indefinitely. To generalize this notion, consider the sum

$$s_n = u_1 + u_2 + \cdots + u_n,$$

formed by taking the first n terms of the infinite series. If the number of terms in s_n is increased indefinitely, one of three things will be true:

(1) The sum s_n approaches a limit which is finite and of course definite; for example, the limit of the sum of the first n terms of the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots$$

of the stick problem is 2.

(2) The sum s_n will increase without limit, as in the case of the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\cdots.$$

To see that the sum of n terms of this series will increase without limit, we have only to note that the terms after the second may be grouped by taking the third and fourth, the fifth to the eighth, the ninth to the sixteenth, and so on, and in each case the sum of the terms in a group is greater than $\frac{1}{2}$. The process of grouping may be continued indefinitely, so that the series will appear as the sum of an infinite number of finite terms.

(3) The sum s_n will approach no limit, either finite or infinite. Such a series is

$$1-1+1-1+1-\cdots,$$

where the value of s_n is always 1 or 0.

If the sum s_n approaches a finite limit S, as in the first case, the series is said to be *convergent* with sum S, and we write

$$S=u_1+u_2+u_3+\cdots.$$

In all other cases the series is said to be *divergent.* A divergent series has no value whatever for the purposes of ordinary computation; whence it follows that the prime question for consideration is the determination of the character of the series as to convergence or divergence.

A *necessary* condition for the convergence of an infinite series is that the successive terms themselves should diminish toward zero, for if we denote the sum of all the terms after the nth by R_n, we shall have

$$S=s_n+R_n.$$

Since

$$\lim_{n=\infty} s_n=S,$$

it follows that

$$\lim_{n=\infty} R_n=0.$$

From this it follows that the sum of any number of terms following the nth approaches zero as n increases, and in particular the limit toward which the value of a single term a_n approaches as n increases is zero.

A single example, viz. the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots,$$

which was shown to be divergent, shows that the condition is not *sufficient* to insure convergence.

No general practical test for the determination of convergence or divergence is known, but there are two, the *direct-comparison test* and the *test-ratio test,* which suffice for the determination of the character of a large number of important series.

In what follows only series with real terms will be considered.

6. The direct-comparison test. In the direct-comparison test the terms of the series to be tested are compared with the corresponding terms of a series whose character is known. This test,

when used to determine the character of series having only positive terms, may be stated in the form of the following

THEOREM. *Let* $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ *be a series of positive terms which is to be tested for convergence or divergence, and let* $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ *be a test series of positive terms whose character is known. If the test series is convergent, and if the terms of the series to be tested are less than, or at most equal to, the corresponding terms of the test series, the series to be tested is convergent, and its sum is not greater than the sum of the test series; if, on the other hand, the test series is divergent, and if the terms of the series to be tested are equal to or greater than the corresponding terms of the test series, the series to be tested is divergent.*

Proof when the test series is convergent. Let

$$a_1 + a_2 + \cdots + a_n + \cdots$$

be a series of positive terms which is known to be convergent with sum A, and let A_n be the sum of the first n terms. Also let

$$u_1 + u_2 + \cdots + u_n + \cdots$$

be a series of positive terms to be tested, and let s_n be the sum of the first n terms. We have to prove that if we have always

$$u_n \leqq a_n,$$

s_n approaches a limit which is equal to or less than A.

Since every term u is less than, or at most equal to, the corresponding term a, the sum

$$s_n = u_1 + u_2 + \cdots + u_n$$

will be less than, or at most equal to, the sum

$$A_n = a_1 + a_2 + \cdots + a_n.$$

But by hypothesis $\lim\limits_{n=\infty} A_n = A$

and, since the terms of the test series are all positive,

$$A_n < A$$

for every value of n. Consequently, the variable s_n is always less than A. Moreover, because the series to be tested contains nothing but positive terms, s_n always increases as n increases.

It follows that s_n is a variable which always increases but never takes values greater than a given fixed number A. By Theorem I (§ 4) s_n approaches a limit which is equal to or less than A, and the series to be tested is convergent.

Proof when the test series is divergent. In this case $s_n \equiv A_n$, and A_n increases without limit. Obviously, s_n, which is not less than A_n, will also increase without limit. Consequently, the series to be tested is divergent.

ILLUSTRATIVE EXAMPLE. It is required to determine the character of the series

$$\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots+\frac{1}{1\cdot 2\cdot 3\cdots n}+\cdots.$$

For a comparison series, the geometrical series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots,$$

whose ratio is $\frac{1}{2}$, and which is therefore convergent, may be taken. Each of the first two terms of the series to be tested is equal to the corresponding term of the test series, as may be seen by inspection. Moreover, every term of the series to be tested, after the second, is less than the corresponding term of the test series, since

$$\frac{1}{1\cdot 2\cdot 3\cdot 4\cdots n}<\frac{1}{2^{n-1}}$$

for all values of n greater than 2. The series to be tested is therefore convergent.

The direct-comparison test is frequently useful in cases where the conditions of the theorem do not hold for the first few terms of the series to be tested and the test series. For example, comparing the series

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

with the convergent series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots,$$

it is seen that the condition required by the test holds for the first term but does not hold for the next three. If, however, we strike out the first term of the series to be tested, or if we prefix the term 2 to the test series, the difficulty disappears, while the character of both the series in question remains unchanged.

In general, a finite number of terms may be stricken out from the beginning of a series with constant terms, and the test applied to the resulting series, without any loss of generality, for the sum of the terms stricken out is finite and determinate. Consequently, if the sum of the first n terms remaining after terms are stricken out from the beginning has a limit as n increases, the sum of the first n terms of the original series will have a limit as n increases. This last limit will be the sum of the terms stricken out plus the sum of the convergent series that remains after the terms are stricken out.

For the purposes of the direct-comparison test it is desirable to have a number of series known to be convergent or divergent to begin with. All geometrical series whose ratio is less than 1 are convergent, while all geometrical series whose ratio is greater than 1 are divergent. The series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

was shown in § 5 to be divergent.

The series $$\frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots$$

is convergent when $k > 1$ and divergent when $k < 1$. The proof is given in most textbooks on algebra.

7. The test-ratio test. The test-ratio test, which is one of the most useful known tests, holds for series having both positive and negative terms. It makes use of the ratio of the general term to the preceding term, viz. $\frac{u_{n+1}}{u_n}$, the so-called *test ratio.* The test-ratio test is given in one of its most useful forms by the following

THEOREM. *Let* $$u_1 + u_2 + u_3 + \cdots$$
be an infinite series, and let

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = t;$$

then, *if* $|t|$* < 1, *the series converges;*
if $|t| > 1$, *the series diverges;*
if $|t| = 1$, *the character of the series is undetermined.*

* As in § 4, the notation $|t|$ is used to denote the numerical value of t.

Proof for series having only positive terms. By the conditions of the theorem

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = t,$$

and for the case under consideration t is a positive number.

Suppose first that

$$t < 1.$$

It is then possible to find a fixed number r lying between t and 1, i.e. a number satisfying the condition

$$t < r < 1.$$

A set of constant numbers

$$k_1, \quad k_2, \quad k_3, \quad \cdots$$

can then be found such that

$$u_1 = k_1 r, \quad u_2 = k_2 r^2, \quad u_3 = k_3 r^3, \quad \cdots,$$

and the series itself may be written

$$k_1 r + k_2 r^2 + k_3 r^3 + \cdots.$$

It follows that after a definite term for which $n = m - 1$,

$$k_m \overset{>}{\underset{\not<}{<}} k_{m+1} \overset{>}{\underset{\not<}{<}} k_{m+2} \cdots.$$

For,

$$\frac{u_{n+1}}{u_n} = \frac{k_{n+1} r^{n+1}}{k_n r^n},$$

and, moreover, because the variable $\frac{u_{n+1}}{u_n}$ approaches the limit t, the value of $\frac{u_{m+1}}{u_m}$ will be less than r, so that

$$\frac{k_{m+1} r^{m+1}}{k_m r^m} = \frac{k_{m+1} r}{k_m} < \cancel{t}\, r \qquad \therefore \quad \frac{k_{m+1}}{k_m} < 1$$

and, from the inequality,

$$\frac{k_{m+1}}{k_m} < \frac{t}{r}.$$

Now, since the fraction $\frac{k_{m+1}}{k_m}$ is less than the proper fraction $\frac{t}{r}$, the numerator k_{m+1} must be less than the denominator k_m, as was asserted.

Furthermore, since $k_m < k_{m+1} < k_{m+2} < \cdots,$

the terms of the series

$$u_m + u_{m+1} + u_{m+2} + \cdots,$$

or, what is the same thing, the terms of the series

$$k_m r^m + k_{m+1} r^{m+1} + k_{m+2} r^{m+2} + \cdots,$$

are, after the first term, less than the corresponding terms of the geometrical series

$$k_m r^m + k_m r^{m+1} + k_m r^{m+2} + \cdots,$$

which is convergent, since the ratio r is numerically smaller than unity. Consequently, the series

$$u_1 + u_2 + u_3 + \cdots,$$

from which the convergent series

$$u_m + u_{m+1} + u_{m+2} + \cdots$$

was obtained by striking out the first $m-1$ terms, is convergent.

In a similar manner, when $t > 1$, it may be shown that after a certain term $u_{m'-1}$ of the series

$$u_1 + u_2 + u_3 + \cdots$$

the terms $\qquad u_{m'}, \quad u_{m'+1}, \quad u_{m'+2}, \quad \cdots$

will be equal to or greater than the terms of the divergent geometrical series $\quad k_{m'} r^{m'} + k_{m'} r^{m'+1} + \cdots.$

Under such conditions the series

$$u_{m'} + u_{m'+1} + u_{m'+2} + \cdots,$$

and consequently the series

$$u_1 + u_2 + u_3 + \cdots,$$

from which it was obtained by striking out $m'-1$ terms, will be divergent.

Finally, if $t = 1$, no conclusion can be drawn, for the limit of the test ratio of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

which was shown to be divergent, is unity, while, on the other hand, the limit of the test ratio of the convergent series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

is also unity.

The theorem is therefore proved for series having only positive terms. The proof for series having both positive and negative terms is not difficult but cannot well be given here.

ILLUSTRATIVE EXAMPLE. It is required to determine the character of the series

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.$$

The general term is $\frac{1}{1\cdot 2\cdot 3\cdots n}$, and the ratio $\frac{u_{n+1}}{u_n}$ is therefore

$$\frac{\frac{1}{1\cdot 2\cdot 3\cdots n\cdot(n+1)}}{\frac{1}{1\cdot 2\cdot 3\cdots n}}=\frac{1}{n+1}.$$

Consequently,

$$\lim_{n=\infty}\frac{u_{n+1}}{u_n}=t=\lim_{n=\infty}\frac{1}{n+1}=0.$$

Since $t=0$, the series is convergent.

EXAMPLES

Determine the character of the following series:

1. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\cdots.$

2. $1+\frac{1}{1^2\cdot 2^2}+\frac{1}{2^2\cdot 3^2}+\frac{1}{3^2\cdot 4^2}.$

3. $\frac{3}{4}+\frac{3\cdot 5}{4\cdot 7}+\frac{3\cdot 5\cdot 7}{4\cdot 7\cdot 10}+\cdots+\frac{3\cdot 5\cdot 7\cdots(2n+1)}{4\cdot 7\cdot 10\cdots(3n+1)}.$

4. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots.$

6. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots.$

7. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots.$

8. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots.$

8. Power series. In most of the series studied so far the terms were constant. The most important series, however, are those in which the terms contain some variable, and of these the most important are the so-called *power series.* A power series is a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

i.e. a series whose terms are positive integral powers of a variable, each multiplied by a constant, and in which the terms are arranged according to ascending powers of the variable. The tests obtained for series with constant terms apply to power series also, but it is important to notice that in general the character of the series depends upon the value that is given to the variable. Indeed, the problem in the case of power series usually takes the form, to find those values of the variable for which the series will be convergent. For example, we know that the series

$$1 + x + x^2 + x^3 + \cdots$$

is convergent for all values of x satisfying the condition

$$-1 < x < +1, \text{ or, briefly, } |x| < 1.$$

We have not ascertained its character when $x = 1$ or $x = -1$, but for all values of x which are less than -1, or greater than $+1$, the series is divergent.

One great value of power series lies in the fact that they furnish an easy means of computing the numerical values corresponding to special values of the variable for many functions where such computation would otherwise be extremely difficult or even impossible. For example, the cosine of x may be defined by the series

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \cdots,$$

where the unit of measure for x is the radian which is the equivalent of $57^\circ.2957795+$. Suppose it is required to find the cosine of one radian. By definition,

$$\cos(1) = 1 - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - .$$

Five terms of this series give

$$\cos(1) = .540302,$$

which is correct to the fifth decimal place. Further examples are given at the end of the chapter.

Among the most important power series are the binomial series, the exponential series, and the logarithmic series. These three will be considered in the order named.

9. The binomial series. In elementary algebra we are made familiar with the expansion of a power of a binomial when the exponent is a positive integer. Such an expansion has the form

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \cdots + x^n. \qquad (1)$$

This expansion is a power series consisting of a finite number of terms with coefficients

$$a^n, \quad \frac{n}{1} a^{n-1}, \quad \frac{n(n-1)}{1 \cdot 2} a^{n-2}, \quad \cdots, \quad 1,$$

and whose general term is

$$\frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} \cdot a^{n-r}x^r. \qquad (2)$$

It may also be looked upon as a power series all of whose coefficients after the $(r+1)$st are zero. If the exponent n is not a positive integer, it is still possible to obtain a development for $(a+x)^n$ which will be similar in form to (1) and for which the general term will be (2). There is this great difference, however: when n is not a positive integer, the series will never end but is a true infinite series.

The power series

$$a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \cdots \qquad (3)$$

is called the *binomial series.*

To determine the character of the binomial series, consider the test ratio, viz.

$$\frac{\dfrac{n(n-1)(n-2) \cdots (n-r+1)}{1\ 2 \cdot 3 \cdots (r-1)r} a^{n-r}x^r}{\dfrac{n(n-1)(n-2) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} a^{n-r+1}x^{r-1}} = \frac{n-r+1}{r} \cdot \frac{x}{a}.$$

The limit of the test ratio is

$$\begin{aligned} t &= \lim_{r=\infty} \frac{n-r+1}{r} \cdot \frac{x}{a} \\ &= \frac{x}{a} \lim_{r=\infty} \frac{n-r+1}{r} \cdot \frac{x}{a} \\ &= -\frac{x}{a}. \end{aligned}$$

But in order that the series (3) should be convergent, t must be less in absolute value than unity, i.e.

$$\left| -\frac{x}{a} \right| < 1, \quad \text{or} \quad |x| < |a|. \tag{4}$$

If, on the other hand, $|x| > |a|$,
the series (3) will be divergent.

We write

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \cdots$$

for all values of n, with the understanding that the power series represents the development of $(a+x)^n$ for those values of x which satisfy the convergence condition (4).

A case of special importance is that for which $a = 1$; then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \cdots \tag{5}$$

for values of x such that $|x| < 1$.

The series (5) is of great use in computing many quantities which occur in the theory of interest. As a simple example, suppose the value of the expression $\frac{1}{1.04}$ is required. This value may readily be obtained by replacing x by .04 in the binomial expansion,

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots. \tag{6}$$

The result is

$$\begin{aligned} \frac{1}{1.04} = (1.04)^{-1} &= 1 - .04 + (.04)^2 - (.04)^3 + \cdots \\ &= 1.0016 - .040064 \\ &= .961536 +. \end{aligned}$$

This result, which is obtained by taking four terms of the series, is true to the fifth decimal place.

Another expression which could not be easily computed by ordinary means, and which occurs frequently in practice, is $(1.04)^{\frac{1}{12}}$. By the binomial series

$$(1.04)^{\frac{1}{12}}=1+\frac{1}{12}.04+\frac{\frac{1}{12}\left(\frac{1}{12}-1\right)}{1\cdot 2}(.04)^2$$
$$+\frac{\frac{1}{12}\left(\frac{1}{12}-1\right)\left(\frac{1}{12}-2\right)}{1\cdot 2\cdot 3}(.04)^3+\cdots.$$

Four terms of the series give the result

$$(1.04)^{\frac{1}{12}}=1.003273+,$$

which is accurate to the sixth decimal place.

10. The exponential series. The expression

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n \qquad (1)$$

occurs so frequently and is of such importance in mathematics that it may well be called a fundamental limit.

To find an expression for it that will be useful in practical work, suppose that n approaches infinity through the series of natural numbers, i.e. through the positive integers taken in order of increasing magnitude. For any positive integral value of n

$$\left(1+\frac{x}{n}\right)^n=1+n\frac{x}{n}+\frac{n(n-1)}{1\cdot 2}\frac{x^2}{n^2}$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{x^3}{n^3}+\cdots\frac{x^n}{n^n}. \qquad (2)$$

To find the value of the limit (1) by roughly approximate methods, divide out the n's from the numerators and denominators, and (3) takes the form

$$\left(1+\frac{x}{n}\right)^n=1+\frac{x}{1}+\frac{1-\frac{1}{n}}{1\cdot 2}x^2$$
$$+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\cdot 2\cdot 3}x^3+\cdots \text{ to } n+1 \text{ terms.} \qquad (3)$$

If we assume that the limit of the sum on the right in (3) can be found by taking the sum of the limits of the separate terms, even though the number of terms be indefinitely increased, we shall have

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n=1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots. \tag{4}$$

The power series on the right is called the *exponential series* and is ordinarily denoted by e^x. By definition of e^x,

$$e^x=1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots. \tag{5}$$

The series (5) is convergent for all finite values of the variable x, for the general term is

$$\frac{x^n}{1\cdot 2\cdot 3\cdots n}$$

and the test ratio is

$$\frac{\dfrac{x^{n+1}}{1\cdot 2\cdot 3\cdots n\cdot(n+1)}}{\dfrac{x^n}{1\cdot 2\cdot 3\cdots n}}=\frac{x}{n+1}.$$

Consequently, $t=\lim\limits_{n=\infty}\dfrac{x}{n+1}=x\cdot\lim\limits_{n=\infty}\dfrac{1}{n+1}=x\cdot 0.$

The number t is therefore zero for every finite value of x, and the statement is proved.

It is not difficult to prove that the expression e^x behaves exactly like an ordinary power when combined with other expressions of the same sort. In particular,

$$e^x\cdot e^y=e^{x+y},\quad e^x\div e^y=e^{x-y},\quad\text{and}\quad (e^x)^y=e^{xy}.$$

To find the value of e^1 or e, substitute 1 for x in the series (5). Then

$$e=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.$$

Ten terms of the series will give

$$e=2.71827,$$

accurate to the fifth decimal place. The student is advised to carry out the computation.

11. The logarithmic series. The series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \tag{1}$$

is called the *logarithmic series.* It is convergent for values of x which are numerically less than unity, for the general term is

$$\frac{x^n}{n}$$

and the test ratio is $$\frac{x^{n+1}}{n+1}\div\frac{x^n}{n}=\frac{n}{n+1}\cdot x.$$

The limiting value of the test ratio is then

$$t=\lim_{n=\infty}\frac{n}{n+1}x$$
$$=x\lim_{n=\infty}\frac{1}{1+\frac{1}{n}}=x.$$

The series (1) is therefore convergent for

$$|t|=|x|<1,$$

which was to be proved.

The series (1) is called the logarithmic series because for values of x numerically less than 1, i.e. for values for which the series is convergent, it represents the value of the expression $\log(1+x)$. By definition *

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots. \tag{2}$$

The logarithmic series is not well adapted to computation, since it converges very slowly and a large number of terms are required to obtain a good approximation. The actual work of computing logarithms is best accomplished by the use of some such series as that of Example 9 below, which is derived from the logarithmic series.

* The logarithm that is here defined is the hyperbolic, or Napierian, logarithm. For a definition of different systems of logarithms, see the next chapter.

EXAMPLES

1. Find the value of $(1.05)^8$ to five decimal places.

2. Find the fifth root of 1.05, i.e. the value of $(1+.05)^{\frac{1}{5}}$, to five decimal places.

3. Find the value of e^{-1} to five decimal places by means of the power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots.$$

4. If sin x is defined by the series

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots,$$

find the value of sin (.1) to the fourth decimal place.

5. For what values of x is the series

$$\frac{1}{1\cdot 2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{4\cdot 5} + \cdots + \frac{x^n}{(n+1)(n+2)} + \cdots$$

convergent?

6. For what values of x is the series

$$\frac{1}{1} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \cdots$$

convergent?

7. Prove by actual multiplication that the product of

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

and

$$e^y = 1 + \frac{y}{1} + \frac{y^2}{1\cdot 2} + \frac{y^3}{1\cdot 2\cdot 3} + \cdots$$

is e^{x+y}.

8. Compute the value of e^3 to the fifth decimal place.

9. Compute the value of the logarithm of 2 from the series

$$\log\frac{M}{N} = 2\left[\frac{M-N}{M+N} + \frac{1}{3}\left(\frac{M-N}{M+N}\right)^3 + \frac{1}{5}\left(\frac{M-N}{M+N}\right)^5 + \cdots\right]$$

by making $M = 2$ and $N = 1$.

CHAPTER III

LOGARITHMS

12. Definitions and preliminary notions. In elementary algebra the power a^x is defined for positive, negative, zero, and fractional values of x. If

$$a^x = y, \qquad (A)$$

the number a is called a *base*, y is a number, and x is called the *logarithm* of y to the base a. The logarithm of a number to a given base is defined to be *the exponent which indicates the power to which the base must be raised to produce the given number.* We write

$$x = \log_a y, \qquad (B)$$

so that equations (A) and (B) are only different ways of saying the same thing.

It can be shown that when $a > 0$ and $a \neq 1$, a single real positive value of y exists for every real value of x, even though x be an incommensurable number, like $\sqrt{2}$; and, conversely, for every real positive value of y a single real value, which may be positive or negative, exists for x. In other words, given a positive base a, different from unity, corresponding to every real logarithm, there exists a single positive number; and, conversely, for every real positive number there exists a single real logarithm.

Negative numbers have no real logarithms, for the base a is always taken as a positive number, and the negative numbers that might otherwise exist, as in an equation like $10^{\frac{1}{2}} = y$, have been deliberately excluded from the definition of the expression a^x.

A set of logarithms x of all numbers y, corresponding to a given base a, is called a *system of logarithms.* Two systems are in use, the common, or Briggsian, system, which is used in practical computations, and the hyperbolic, or Napierian, system, which is always used in analytical work. The base of the common system is 10, while the base of the Napierian system is the number $e = 2.71827+$, which was found in § 10.

The following theorems hold for all systems of logarithms:

I. *The logarithm of* 1 *is zero.*

For, suppose $\log_a 1 = x$. This fact may be written $a^x = 1$. But when a is different from unity, the only power of a that can be equal to 1 is a^0.

$$\therefore \log_a 1 = 0.$$

II. *The logarithm of the base is* 1.

For, let $\log_a a = x$. Putting this statement in the form of equation (A),

$$a^x = a,$$

and $x = 1$ is the only real value of x which will satisfy this equation.

III. *The logarithm of a product is equal to the sum of the logarithms of the factors.*

For, let x and y be the factors of a product and m and n their respective logarithms. Then

$$\log_a x = m \quad \text{and} \quad \log_a y = n,$$

or

$$x = a^m \quad \text{and} \quad y = a^n.$$

Consequently,

$$xy = a^{m+n}$$

and

$$\log_a xy = m + n = \log_a x + \log_a y.$$

IV. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Let x be the number, n the exponent of the power, and l the logarithm of x. Then we have

$$l = \log_a x, \quad \text{or} \quad a^l = x;$$

whence

$$x^n = a^{nl}$$

and, consequently,

$$\log_a x^n = nl = n \log_a x.$$

COROLLARY. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For the quotient $\frac{x}{y}$ may be written xy^{-1}. We have then

$$\log_a \frac{x}{y} = \log_a xy^{-1} = \log_a x + \log_a y^{-1} = \log_a x - \log_a y.$$

13. The characteristic and the mantissa for a system of logarithms with base 10. A logarithm which is not an integer, like any other similar number, consists of two parts, one a whole number and the other a number which may be expressed, exactly or approximately, by a decimal fraction. Moreover, any logarithm, or for that matter any negative number, may be written in such form that the decimal part is positive. For example, $-.6090 = -1 + .3010$, which may be written $\bar{1}.3010$. Again, $-1.2351 = -2 + .7649 = \bar{2}.7649$. The change is accomplished in every case by adding 1 to the decimal part and subtracting 1 from the integral part.

When a logarithm is written in such form that the decimal part is positive, the decimal part is called the *mantissa* and the integral part the *characteristic.*

In the common system the mantissa is the same for all numbers having the same sequence of figures. For moving the decimal point is equivalent to multiplying or dividing by an integral power of 10. Since 10 is the base, its logarithm is 1 and the logarithm of any integral power of 10 is an integer. By Theorems III and IV, multiplying or dividing a number by an integral power of 10 will increase or diminish the logarithm of the number by an integer, and the characteristic alone will be changed.

The mantissa of a logarithm is found from a table of logarithms prepared in advance.

The characteristic depends upon the position of the decimal point. It is easily found by means of a table showing the logarithms of powers of 10. Such a table may be written in the following form:

$$
\begin{aligned}
10^{-4} &= .0001 \\
10^{-3} &= .001 \\
10^{-2} &= .01 \\
10^{-1} &= .1 \\
10^{0} &= 1 \\
10^{1} &= 10 \\
10^{2} &= 100 \\
10^{3} &= 1000 \\
10^{4} &= 10000
\end{aligned}
$$

A number with one figure to the left of the decimal point is either 0 or a number between 0 and 10, and, from the table, its logarithm must lie between 0 and 1. Such a logarithm will have 0 for its characteristic. Similarly, the logarithm of a number with two figures to the left of the decimal point is either 1 (the logarithm of 10) or it lies between 1 and 2. In either case the characteristic is 1. In general, *the characteristic of the logarithm of a number greater than* 1 *is positive, or zero, and is one less than the number of figures to the left of the decimal point.*

This statement may be turned about to give the

RULE FOR POINTING A NUMBER WITH A GIVEN POSITIVE, OR ZERO, CHARACTERISTIC: *The number of places to the left of the decimal point, in a number whose logarithm has a positive, or zero, characteristic, is one greater than the number of units in the characteristic.*

Similarly, by inspecting that part of the table which gives the negative powers of 10, we see that *the characteristic of the logarithm of a pure decimal is negative and is numerically one greater than the number of zeros immediately following the decimal point.* For example, the number .0035 lies between .001 and .01. Its logarithm, therefore, lies between -3 and -2 and so must be -3 *plus* a positive mantissa.

From the foregoing statement concerning the characteristic of the logarithm of a pure decimal we obtain the

RULE FOR POINTING A NUMBER WHOSE LOGARITHM HAS A NEGATIVE CHARACTERISTIC: *To point off a number whose logarithm has a negative characteristic, place as many zeros between the first significant figure on the left and the decimal point as there are units in the negative characteristic, less one.*

Negative characteristics are usually written with the negative sign above the characteristic instead of in front of it; thus, $\bar{2}.3010$. A still better method is to express the negative characteristic as the difference between a positive integer and 10, writing the -10 at the end. Thus, we would write $8.3010-10$ rather than $\bar{2}.3010$.

To facilitate division of logarithms, we may increase both the positive and the negative parts of the characteristic by such a

multiple of 10 that the quotient of the negative part by the divisor is 10. Thus, if we wish to divide $8.3010-10$ by 7, we add and subtract 60. The result will then be found as follows:

$$(8.3010-10) \div 7 = (68.3010-70) \div 7 = 9.7573-10.$$

14. Computation by logarithms. The method of computation by logarithms can best be explained by an example.

Let it be required to find the value of the expression

$$\frac{\sqrt{235} \times \sqrt[3]{.01456}}{2534} = M.$$

We have directly

$$\log M = \tfrac{1}{2} \log 235 + \tfrac{1}{3} \log .01456 - \log 2534.$$

Using a four-place table of logarithms, we have

$$\begin{aligned} \tfrac{1}{2} \log 235 &= 1.1855 \\ \tfrac{1}{3} \log .01456 &= \underline{9.3877-10} \\ & 10.5732-10 \\ \log 2534 &= \underline{3.4038} \\ \log M &= 7.1694-10 \\ M &= .001477 \end{aligned}$$

In the above computation we find directly from the table that the logarithm of 235 is 2.3611. To find the logarithm of .01456 we proceed as though we wished to find log 145.6. From the table we find the mantissas of log 145 and log 146 to be .1614 and .1644, with a difference of .0030. Adding six tenths of this difference to .1614, we find .1632 as the required mantissa, so that, by the rule for characteristics,

$$\log .01456 = 8.1632-10.$$

Taking one third of this result after adding and subtracting 20 we find

$$\tfrac{1}{3} \log .01456 = 9.3877-10.$$

To find the logarithm of 2534 we proceed as though we wished to find the mantissa of the logarithm of 253.4.

After we have found $\log M = 7.1694-10$, we look in the table for the mantissa that lies nearest .1694 and below it. We find

.1673. The difference between this mantissa and the next higher is .0030, while the difference between it and .1694 is .0021. We have, then, to increase the number 147, which corresponds to the mantissa .1673, by $\frac{21}{30}$ of 1, or .7. We find, then, that .1694 is the mantissa for 147.7 and, by the rule for pointing,

$$M = .001477.$$

The accuracy of results obtained by means of logarithms depends upon the number of decimal places given in the tables that are used, and this accuracy has reference to the significant figures counted from the left. In general, a table will give trustworthy results in as many significant figures, counted from the left, as there are decimal places given in the logarithms. For example, four-place logarithms would show no difference between 35492367 and 35490000, while, on the other hand, a result like .00003459 would be considered accurate to the eighth decimal place.

Logarithms cannot be used to any great extent in financial computations where large sums are involved. For example, it would take a nine-place table to yield exact results if the sums involved should reach a million dollars.

EXAMPLES

Find by logarithms the values of the expressions in Examples 1–3 and state the degree of accuracy in each case.

1. $\dfrac{\sqrt{293}\cdot(.034)^5}{2345}$. **2.** $\dfrac{\sqrt[5]{2356}\cdot\sqrt[3]{.03426}}{(.02345)^3}$. **3.** $\left(\dfrac{(35.62)^2\cdot(.004593)^5}{(29342)^{\frac{1}{3}}}\right)$.

4. By logarithms find an approximate value for the twentieth term of the series

$$1 + \tfrac{2}{3} + \tfrac{4}{9} + \cdots.$$

5. Find the simple interest on $6237.43 for 7 years and 3 months at 5.632 per cent. Assuming that the work has been done with four-place logarithms, is the result sufficiently accurate for commercial purposes? Explain.

6. Find the volume of a right circular cone whose base is 6 feet in diameter and whose altitude is 16 feet, assuming the formula $\frac{1}{3}\pi r^2 h$ for volume.

7. Find the value of a solid sphere of copper 16 inches in diameter at 19.75 cents per pound, knowing that the specific gravity of copper is 8.838 and that a cubic foot of water weighs 62.5 pounds.

15. Exponential equations. In most of the equations that occur in elementary algebra the exponents are known and the number affected by the exponent is unknown; for example,

$$x^2 + 5x + 6 = 0.$$

It was noted in § 2 that when the number of terms of a geometrical progression is unknown, it is ordinarily impossible to find it by elementary means, since it occurs as an exponent. For example, if the first term, the last term, and the ratio of a geometrical progression are 5, 98415, and 3, respectively, equation (C) of § 2 gives

$$98415 = 5(3)^{n-1},$$

where n, the unknown, is a part of the exponent. This equation is of the same type as the equation

$$a^x = y$$

and may be treated in the same way.

An equation in which the unknown number occurs as an exponent is called an *exponential equation.*

The simplest form of an exponential equation is given by the equation

$$a^x = b. \tag{1}$$

This equation can be solved readily by taking logarithms of both sides. By the theorem for the logarithm of a power,

$$x \log a = \log b;$$

whence

$$x = \frac{\log b}{\log a}. \tag{2}$$

It must be carefully noted that the expression on the right side of equation (2) is the quotient of two logarithms and not the logarithm of a quotient, so that it cannot be written as $\log b - \log a$.

No general rule for the solution of exponential equations can be given, but the solutions of two of the following examples will serve to illustrate the method to be used in the simpler cases.

EXAMPLES

1. Solve the equation $2^x = 3$.

Solution. Taking logarithms of both sides, we find

$$x \log 2 = \log 3,$$

and solving this linear equation for x,

$$x = \frac{\log 3}{\log 2};$$

or, using a four-place table, $$x = \frac{.4771}{.3010}.$$

By actual division (which may of course be performed by logarithms)

$$x = 1.585 +.$$

2. Solve $37^{x+2} \times 15^{x-3} = 14^{2x+1}$ for x.

Solution. Taking logarithms of both sides, we have

$$(x+2)\log 37 + (x-3)\log 15 = (2x+1)\log 14,$$

or $$x\log 37 + x \log 15 - 2x\log 14 = -2\log 37 + 3\log 15 + \log 14.$$

Solving for x, we find
$$\begin{aligned} x &= \frac{-2\log 37 + 3\log 15 + \log 14}{\log 37 + \log 15 - 2\log 14} \\ &= \frac{-3.1364 + 3.5283 + 1.1461}{1.5682 + 1.1761 - 2.2922} \\ &= \frac{1.5380}{.4521} \\ &= 3.402 +. \end{aligned}$$

3. Find x from the equation $30^x = 3000$.

4. Given $\dfrac{30^x \cdot 42^x}{17^x} = 29$; find x.

5. Given $2^x \cdot 3^y = 2000$ and $3y = 5x$; find x and y.

6. Given $l = ar^{n-1}$; find n.

7. Given the two equations
$$l = ar^{n-1},$$
$$s = \frac{rl - a}{r - 1};$$
find n in terms of a, l, and s.

8. Find the expression for n from the equation $s = P(1+i)^n$.

9. Find n from the equation $$A = \frac{(1+i)^n - 1}{i}.$$

16. The transformation of logarithms of one system into logarithms of another system. In theoretical work the so-called Napierian logarithms, whose base is the number $e = 2.71828+$, are almost invariably used, while for numerical computation the common logarithms, whose base is 10, are used. It is therefore frequently necessary to transform a Napierian logarithm into a common logarithm, and vice versa. This transformation is a special case of a more general problem, viz. *to express a logarithm with a given base a in terms of a logarithm with another base b.*

Let N be the number whose logarithm is sought and l its logarithm to the base a. Then, by hypothesis,

$$\log_a N = l. \tag{1}$$

Changing equation (1) to the exponential form,

$$N = a^l. \tag{2}$$

Taking logarithms to the base b of both sides of equation (2),

$$\log_b N = l \log_b a. \tag{3}$$

If l be replaced by its value as given by equation (1), the result is

$$\log_b N = \log_b a \cdot \log_a N, \tag{4}$$

which is the fundamental formula in the problem of transforming the logarithms of one system into those of another system.

If it be required to find the Napierian logarithm when the common logarithm is given, we replace a by 10 and b by $e = 2.71828+$. Equation (4) then becomes

$$\log_e N = \log_e 10 \cdot \log_{10} N. \tag{5}$$

To find the common logarithm when the Napierian logarithm is given, solve (5) for $\log_{10} N$ and obtain

$$\log_{10} N = \frac{1}{\log_e 10} \cdot \log_e N. \tag{6}$$

The factor $\frac{1}{\log_e 10}$, by means of which Napierian logarithms are converted into common logarithms, is called the *modulus of the common system* and is usually denoted by the symbol M, so that by definition

$$M = \frac{1}{\log_e 10}. \tag{7}$$

The number M and its reciprocal,

$$\frac{1}{M} = \log_e 10, \tag{8}$$

have been carefully computed to a large number of decimal places. Their values to the sixth place are

$$M = .434294$$

and

$$\frac{1}{M} = 2.302585.$$

If the values of M and $\frac{1}{M}$ be introduced into equations (5) and (6), these equations become

$$\log_e N = \frac{1}{M}\log_{10} N = 2.302585 \log_{10} N \tag{5'}$$

and

$$\log_{10} N = M \log_e N = .434294 \log_e N. \tag{6'}$$

From equations (5′) and (6′) the following rules for the transformation of logarithms of the one system into logarithms of the other system are easily obtained.

To find the Napierian logarithm when the common logarithm is given, multiply the common logarithm by 2.302585.

To find the common logarithm when the Napierian logarithm is given, multiply the Napierian logarithm by .434294.

EXAMPLES

1. The common logarithm of 2 is approximately .3010300. What is its Napierian logarithm?

2. By means of a table of common logarithms and one of the formulas of the present section find the Napierian logarithm of 258.

3. Prove that $\log_{10} e = \dfrac{1}{\log_e 10}$ and that, in general, $\log_b a = \dfrac{1}{\log_a b}\cdot$ Why do we write $M = \dfrac{1}{\log_e 10}$ rather than $M = \log_{10} e$?

4. Compute the common logarithm of 1.1 to five places by starting from the power series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

CHAPTER IV

GRAPHICAL REPRESENTATION

17. Definitions and first principles. Mathematical relations are frequently made clearer when exhibited to the eye by means of a properly constructed diagram. The method of representing mathematical relations by diagrams is called *graphical representation.*

The simplest graphical representation is the representation of a single number, such as the measure of a length or of a sum of money, along a straight line. Such a representation is accomplished if, with an arbitrary unit of measure, we lay off from an arbitrary point in a straight line a length whose measure is the same as that of the number we seek to represent. For example, five years may be graphically represented by a line five inches long. It should be noted that the same line may just as well be used to represent five million years. If we wish to represent a magnitude capable of taking both positive and negative values, the positive values are laid off to the right of the arbitrary point and the negative values to the left.

The arbitrary line is called an *axis;* the arbitrary point from which measurements are made is called the *origin*, or the *zero point*, and the unit employed is called the *unit of measurement.*

Graphical representation assumes its chief importance in the representation of functions of one or more variables. For present purposes a *function of one variable* may be defined as a mathematical expression which depends for its value upon the value of the variable. For example, if x be a variable, such expressions as x^2, $3x+2$, x^2+4, e^x, ca^x, $\log x$, are functions of x.

A function thus defined is itself a variable whose values are determined when the value of x is known. The variable x is called the *independent variable*, and the function is called the *dependent variable.* A function in which the independent variable is affected by only a finite number of the algebraic operations,

addition, subtraction, multiplication, division, involution, and evolution, is called an *algebraic function*. For example, the function $\frac{3x^2-6}{2x+3}$ is obtained from the variable x by six algebraic operations, viz. squaring x, multiplying by 3, subtracting 6, multiplying x by 2, adding 3, and, finally, dividing $3x^2-6$ by $2x+3$. The functions ax^2+bx+c and $\sqrt{25-x^2}$ are likewise algebraic, while the function

$$e^x = \lim_{n=\infty}\left(1+\frac{x}{n}\right)^n = 1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\cdots,$$

which involves the consideration of an infinite number of algebraic terms, is not algebraic, but belongs to a class known as *transcendental functions*. The function $\log x$ is also transcendental.

Algebraic functions are divided into several classes: *rational functions*, like $\frac{3x^2-6}{2x+3}$, which do not involve root extraction; *rational integral functions*, like $3x^2+5x-6$, which do not involve either division or root extraction; and *irrational functions*, like $\sqrt{25-x^2}$, which involve root extraction.

If it is not necessary to give a function explicitly, or if the character of the dependence is unknown, the symbol $f(x)$ is used to denote a function of x. Different functions are denoted by different letters or by the same letter with different subscripts. If $f(x)=3x^2+6x-5$, the function $3y^2+6y-5$ would be represented by $f(y)$, but the function $2x^2+6x-5$ would have to be represented by a different symbol, say, $f_1(x)$ or $g(x)$.

For the graphical representation of the relation between a variable and a function depending upon it, a second line, called the *function axis*, or the *axis of the dependent variable*, is required. This second axis is drawn through the zero point of the first, at right angles to it, and the intersection of the two axes is taken as the zero point of the second as well as of the first. The two axes are together called *coördinate axes*. If the dependent variable, i.e. the function, be denoted by y, so that by definition

$$y=f(x),$$

we can speak of the axis of the independent variable as the x-axis, and of the function axis as the y-axis. Positive values of

the function are laid off above and negative values below the x-axis, or, to speak more accurately, the positive part of the y-axis has the direction that would be taken by a line which is rotated about the origin in the direction opposite to that described by the hands of a clock, and through an angle of 90° measured from the positive part of the x-axis. A line representing the function is laid off in a line parallel to the y-axis and from the extremity of the line representing the corresponding value of the independent variable. These two lines, representing, respectively, the independent variable and the function, will determine uniquely a point in the plane of the coördinate axes, and, conversely, any point in the plane of the axes determines two lines, viz. the perpendiculars to the two axes.

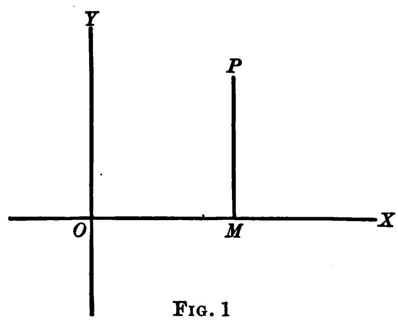

FIG. 1

A point in the plane of coördinate axes may be considered without reference either to function or to variable. The perpendicular from the point to the x-axis is called the *ordinate* of the point, and the distance from the origin to the foot of the ordinate is called the ***abscissa***. The abscissa and the ordinate are together called the ***coördinates*** of the point. In the figure (Fig. 1) the line OM is the abscissa and the line MP the ordinate of the point P. The terms ***abscissa***, ***ordinate***, and ***coördinate*** are used to denote either the lines determined by the point in question or the numbers indicating the lengths of the lines. In the latter case we write (x, y) to denote the coördinates of a point whose abscissa is x and whose ordinate is y.

To return to the graphical representation of a function of a variable, consider the function

$$y = 2x + 1.$$

For any given value of x, say, $x = 2$, the value of y corresponding to it can be computed. When $x = 2$, $y = 5$, and this pair of values determines a definite point in the coördinate plane.

Similarly, for any value of x the corresponding value of y may be found, and the point determined by the pair of values may be located. The totality of all points so determined will lie on a line, either curved or straight, and this line is the *graph* of the function $2x+1$, or, what is the same thing, the graph of the equation

$$y=2x+1.$$

The actual construction of the graph is best accomplished by forming a table of values of the independent variable, together

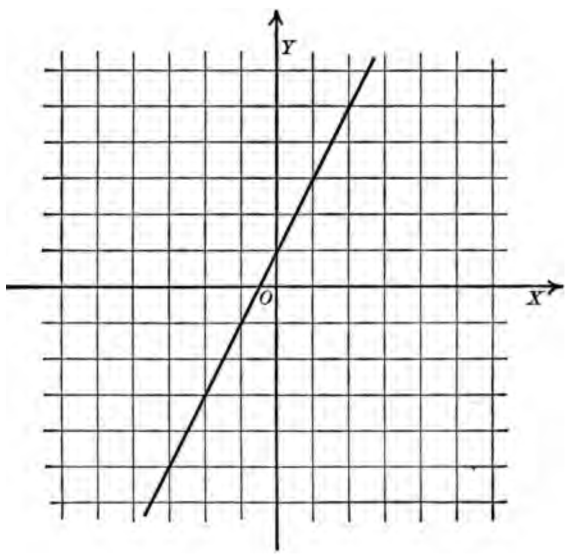

FIG. 2

x	$y=2x+1$
-4	-7
-3	-5
-2	-3
-1	-1
0	$+1$
1	$+3$
2	$+5$
3	$+7$
4	$+9$

with the corresponding values of the function, such as is shown herewith. The table may be extended at will in either direction, or a larger number of values of x and corresponding values of the function may be considered for any given interval. The points determined by each pair of values are then located, and a smooth curve is drawn through the points. In the present example the graph seems to be, as it really is, a straight line (see Fig. 2).

In a similar fashion we may construct the graph of any function whose values corresponding to given values of the independent

variable may be computed. It may be that the function is such that to a given value of the independent variable more than one value of the function, or no value, will correspond. Such a function is

$$y = \pm\sqrt{25 - x^2};$$

for if the number 3 be substituted for x, two values, $+4$ and -4, are found for y. Clearly the substitution for x of any value lying between $+5$ and -5 will give two values for y. On the other hand, if $x = 10$, $y = +\sqrt{-75}$ or $-\sqrt{-75}$, both of which are imaginary and cannot be represented by the method under consideration. The graph has, then, no real existence for values of x which are less than -5 or greater than $+5$. It will be found to be a circle with radius 5, having its center at the origin.

It is not at all necessary that we should know the form of the function, provided we have a means of determining the values corresponding to the values of the independent variable. The function values may be determined, as they frequently are, by observation. Such examples would be the hourly thermometer readings for a day, the population of a city taken annually, the increase of a city's expenses by years. In fact, any set of observed data which varies with the time may be represented graphically by using the time as abscissa and the corresponding observed value as ordinate.

Illustrative Example. On a certain day, at a given station, the hourly thermometer readings, beginning with midnight, were observed to be 0°, $-1°$, $-3°$, $-5°$, $-7°$, $-9°$, $-8°$, $-6°$, $-4°$, $-2°$, 0°, 3°, 7°, 10°, 11°, 11°, 9°, 8°, 7°, 6°, 4°, 2°, 0°, $-2°$, $-4°$. What is the temperature curve for the day?

Solution. On the axis of abscissas lay off the values of the time from one hour to twenty-four, and at the extremities of the abscissas erect perpendiculars representing the observed temperatures. A smooth curve drawn through the extremities of these ordinates will be the curve required. Its form is shown in Fig. 3.

It should be carefully noted that there is a very sharp difference in the character of the two problems just solved. In the first the value of the function $2x + 1$ depended upon the value of the independent variable x, and upon that alone, while in the second the observed temperature depended upon many other

things besides the time, such as the season of the year, the latitude and the altitude of the station, and so on. Indeed, in problems of the second kind the relation of ordinate to abscissa can scarcely be called a functional relation. It is, rather, a mere correlation of values. The curve is not for that reason without value. The method is largely used in the theory of statistics, which is applied to the investigation of a variety of important problems.

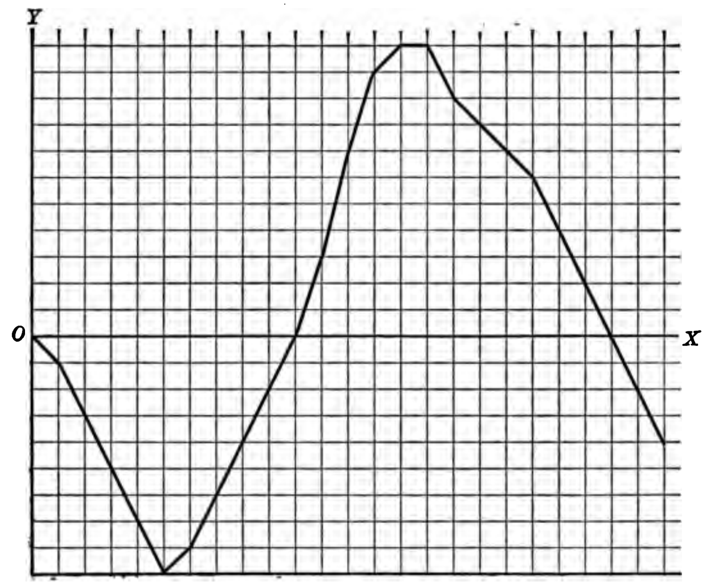

FIG. 3

18. Some important graphs. 1. *The graph of the linear integral function.* A rational integral function (see § 17) in which the variable occurs in the first degree and no higher is called a *linear integral function.* It is necessarily of the form $ax+b$, where a and b are constants, so that

$$y = ax + b \tag{1}$$

is the equation between the function and the independent variable. The reason for the use of the term *linear* is given by the following

THEOREM: *The graph of a linear integral function, or, what amounts to the same thing, the graph of an equation of the first degree between two variables, is a straight line.*

Let P be any point on the graph, B the intersection of the graph with the y-axis, and A the intersection of the graph with the x-axis, as in Fig. 4. Draw the ordinate MP of the point P, and draw a line BR from B parallel to the x-axis and meeting

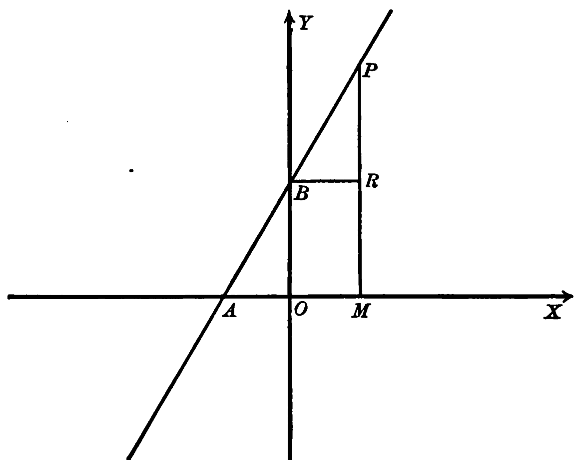

FIG. 4

the ordinate MP in R. The abscissa of the point B is 0. Substituting this value of x in the equation

$$y = ax + b,$$

we find for the value of the ordinate of the point B

$$y = b,$$

i.e. the length of the line OB is b. Moreover,

$$\begin{aligned} PR &= MP - MR = MP - OB \\ &= y - b. \end{aligned}$$

But $BR = OM = x$. Hence the ratio of the two lines RP and BR is

$$\frac{RP}{BR} = \frac{y-b}{x}.$$

But, from equation (1),

$$\frac{y-b}{x} = a, \tag{2}$$

as is easily seen when b is transposed to the left member and the whole equation divided through by x. Consequently,

$$\frac{RP}{BR} = a. \tag{3}$$

This relation may be translated into words by saying that *the ratio of the distances of the point P from two fixed lines, viz. BR and BY, is constant.* From geometry we know that the locus of a point, the ratio of whose distances from two fixed lines is constant, is a straight line. Therefore the graph of the equation

$$y = ax + b,$$

or of the linear function $ax + b$, is a straight line, as the theorem asserts.

Referring to Fig. 4, we see that the two lines BR and RP, which occur in the equation (3), may be likened to the tread and the riser of one of the steps of a stairway. It is obvious that the steepness of the stairway depends upon the ratio of the length of the riser to the length of the tread. If the riser RP is long in comparison with the tread BR, the stairway will be steep, while if it is short in comparison with BR, the slope will be gentle. In other words, the value of a determines the steepness of the line. For this reason a is called the *slope* of the line. The term *slope* is used in exactly the same way as a carpenter uses the term *pitch* when speaking of the steepness of a roof.

When the graph is not a straight line, the slope at any point is defined as the slope of the tangent at that point. The slope of a curved line is not less important than that of a straight line. It is, however, beyond the scope of this book to take up the discussion of the slope of a curve, which is a matter belonging to the differential calculus.

2. *The graph of the quadratic integral function.* A *quadratic integral function* is an algebraic function which is rational, integral, and of the second degree. Its form is

$$ax^2 + bx + c,$$

and it has the same graph as the equation

$$y = ax^2 + bx + c. \tag{4}$$

The graph is a well-known curve called a parabola. The graph of the quadratic function

$$x^2 - 5x + 6$$

is shown in Fig. 5.

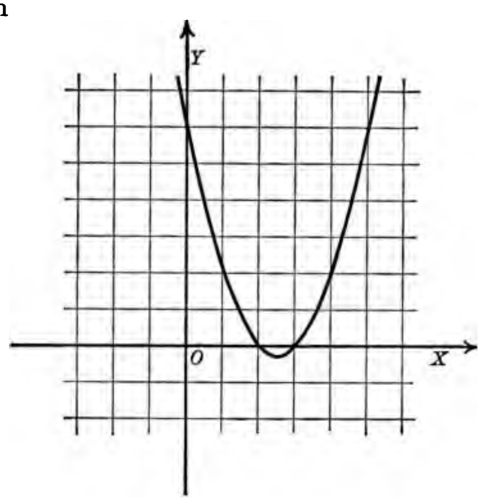

FIG. 5

3. *The logarithmic function.* The graph of the function $\log x$ or the equation

$$y = \log x \quad (5)$$

is nothing more or less than a graphical representation of a portion of a table of logarithms forming a given system. The graph of

$$y = \log_{10} x$$

is given in Fig. 6. No computation is required if a table of common logarithms is at hand, and the construction of the curve presents no difficulty.

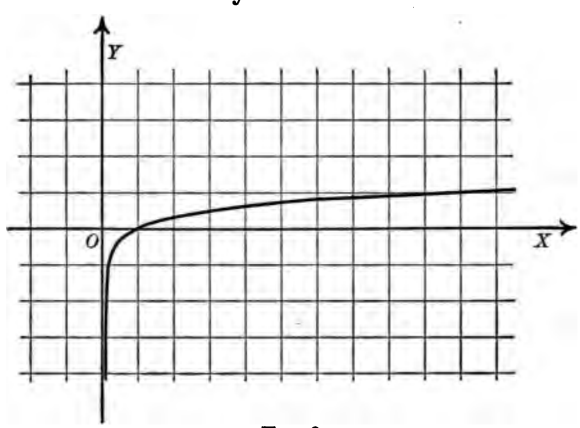

FIG. 6

All logarithmic curves whose equations have the form (5) pass through the point (1, 0). Why is this statement true?

4. *The exponential function.* An *exponential function* is one in which the variable occurs as an exponent. The simplest form of such a function is a^x, though the name, exponential function, is usually reserved for the special function e^x, where e denotes the Napierian base $2.71828 \cdots$.

The table of values is easily constructed for integral values of the exponent when the base is an integer. For $a = 2$ the table

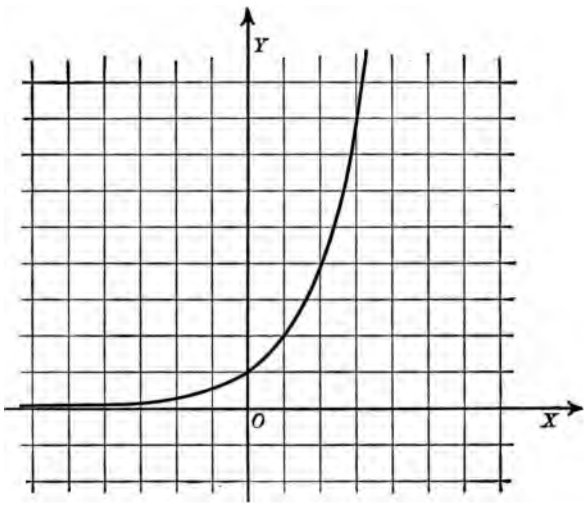

FIG. 7

x	$y = 2^x$
-5	$\frac{1}{32}$
-4	$\frac{1}{16}$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16

of values is as shown in the accompanying table, and the curve, which is the graph of the equation

$$y = 2^x,$$

is that shown in Fig. 7. All exponential curves whose equations are of the type

$$y = a^x \tag{6}$$

pass through the point $(0, 1)$. Why?

There are many phenomena which follow a law whose mathematical statement is equation (6). Such phenomena are said to follow the *compound interest law*, a law which is of very great importance in many branches of pure and of applied mathematics as well.

EXAMPLES

1. Construct the graphs of the functions x, $2x$, $3x$, and $4x$ with the same coördinate axes.

2. Construct the graphs of the functions $2x-1$, $2x$, $2x+1$, and $2x+2$ with the same coördinate axes.

3. Assuming that the graphs in Example 2 are straight lines, prove that they are parallel.

4. Construct the graphs of the equations

$$y = x^2, \quad y = \tfrac{1}{8}(x-2)^2, \quad 8y = x^2 - 5x + 6.$$

5. Construct the graph of $x^2 + y^2 = 25$.

6. Construct the graph of the function $y = x^3$.

7. Beginning with the year 1876, the high-school attendance in this country was 22,982; in 1880 it was 26,609; and at the end of the succeeding ten-year periods up to 1910 it was as follows: in 1890, 202,963; in 1900, 519,251; and in 1910, 915,061. Construct the graph showing the high-school attendance for these years. What was the approximate attendance in 1895? What is it likely to be in 1920?

8. Construct the exponential curve whose equation is $y = 3^x$.

9. From the curve in Example 8 find the logarithm of 10 to the base 3.

10. Construct the curve of the function A when A is given by the equation $A = (1.05)^n$.

11. Construct the graph of the function $100 \times .05 \times t$, and show how it may be used as a table for simple interest at 5%.

12. Construct the graph of the function

$$V^n = 100{,}000\left[\frac{(1.05)^{20} - (1.05)^n}{(1.05)^{20} - 1}\right]$$

where n is variable.

13. If a function $f(x)$, which is continuous between $x = a$ and $x = b$, is negative for the value $x = a$ and positive for $x = b$, at least one root of the equation $f(x) = 0$ lies between $x = a$ and $x = b$. Show that this theorem may be made evident by graphical representation of $f(x)$.

PART II. INTEREST AND ANNUITIES

CHAPTER V

INTEREST

19. Simple interest. Broadly speaking, *interest* is defined as the income on capital profitably invested. In a more restricted sense it is the sum paid for the use of money that is loaned. The sum invested, whether it be in the form of money loaned or capital invested in a business enterprise, is called the *principal.* Interest is computed as so many hundredths of the principal earned in a given unit of time. The unit of time is almost invariably one year. The *rate of interest** is that fraction which expresses the ratio of the interest earned in the unit of time to the principal. It may also be defined as the fraction which expresses the amount paid for the use of a unit of principal for a unit of time. *Simple interest* is interest which is proportional to the time. The sum of the principal and the interest is called the *amount.*

If we use the symbols P, I, i, n, and A, to denote the principal, the interest, the rate, the time, and the amount, respectively, the fundamental formulas for simple interest are

$$I = Pni \tag{1}$$

and

$$A = P + I,$$

or

$$A = P(1 + ni). \tag{2}$$

All problems that can arise in simple interest may be solved by means of formulas (1) and (2).

* Care must be taken to note that the rate of interest is always a fraction. We use the expression 6% meaning that the rate is .06.

ILLUSTRATIVE EXAMPLE. Let it be required to find the principal which will yield \$500 in interest at 5% in 2 years and 6 months.

By formula (1)

$$500 = P \times .05 \times 2.5;$$

whence

$$P = \frac{500}{.05 \times 2.5}$$
$$= \$4000.$$

20. Ordinary and exact interest. In practice the greater number of problems that occur in simple interest are problems involving short periods of time. For reasons of convenience it is customary in ordinary transactions to compute the interest on the basis of 360 instead of 365 days in a year. Interest computed on the basis of 360 days in a year is called *ordinary interest.* Interest computed on the basis of 365 days in a year is called *exact interest.*

To find the relation between the two kinds of interest, let I denote ordinary and I' exact interest. If d denote the time expressed in days,

$$I = Pi\frac{d}{360}, \tag{1}$$

and

$$I' = Pi\frac{d}{365}. \tag{2}$$

To find the relation between I and I', divide equation (1) by equation (2) and obtain

$$\frac{I}{I'} = \frac{365}{360} = 1 + \frac{1}{72};$$

whence

$$\boldsymbol{I = I' + \frac{I'}{72}}. \tag{3}$$

Translating this formula into words, we obtain as the rule for deriving ordinary from exact interest: *Ordinary interest is equal to the exact interest increased by one seventy-second of itself.*

Solving formula (3) for I',

$$I' = \frac{72}{73}I,$$

or,

$$\boldsymbol{I' = I - \frac{I}{73}}. \tag{4}$$

From formula (4) we have the rule for deriving exact interest from ordinary interest: *Exact interest is equal to ordinary interest diminished by one seventy-third of itself.*

ILLUSTRATIVE EXAMPLE. Find the exact interest on \$5278.17 from May 11, 1911, to June 25, 1911, at 6%.

The time is 45 days, which is $\frac{1}{8}$ of 360 days. Hence, the ordinary interest is

$$\$5278.17 \times .06 \times \tfrac{1}{8} = \$39.586.$$

The exact interest is then

$$\$39.586 - \frac{\$39.586}{73} = \$39.044.$$

21. Computation of simple interest. The computation of simple interest, however the work may be arranged, depends upon one of the formulas

$$I = Pni,$$

$$I = Pi\frac{d}{360},$$

and

$$I' = Pi\frac{d}{365}.$$

If a large amount of such work is to be done, an interest table is almost a necessity. Elaborate tables giving the interest on all sums up to \$1000, for times less than one year and at rates from 3 to 8 or 9%, are published for the use of bankers and others who wish to obtain results quickly. For ordinary purposes a much simpler table is sufficient.

From Table II we may find the exact interest at 5%, on all amounts from \$1 to \$100,000, for times up to 365 days, with a small amount of labor. A table like Table I, which gives the number of the day of the year, counting forward from January 1, is a useful aid in finding the time between two dates.

Since simple interest is proportional to the rate, we may find the interest at a rate differing from 5% by increasing or diminishing the interest at 5% by the proper number of fifths of itself. Thus, to find the interest at 6% we add to the interest at 5% one fifth of itself.

EXAMPLES

1. Compute the exact interest on $43,729.29 from April 1 to May 18 at 5%.

The time is 47 days. From the table we find

Interest on	$40000	=	$257.534
Interest on	3000	=	19.315
Interest on	700	=	4.507
Interest on	20	=	.129
Interest on	9	=	.058
Interest on	.29	=	.001
Interest on	$43729.29	=	$281.54

The simple interest on the same principal and for the same time at 6% would be

$$\$281.54 + \$56.31 = \$337.85.$$

2. Find the simple interest on $23,738.42 from February 1, 1908, to March 30, 1908. $$23,738.42 \times .05 \times \frac{58}{360} = 23,738.42 \times \frac{.58}{72} = $191.23

3. Find the simple interest on $38,472.27 from March 21 to April 11 of the same year. 21 days $38,472.27 \times .05 \times \frac{21}{360} = 38,472.27 \times \frac{.21}{72} = $112.21

22. Compound interest. Interest, whether it be in the form of money paid for the use of money that has been loaned, or whether it be in the form of dividends on capital profitably invested, should be paid promptly when due. When paid, it may itself be put at interest, so that it is, in effect, added to the principal to form a new principal. This process may go on to any length. In case the interest is not paid by the borrower, it may still be considered as added to the principal at stated intervals. When the interest is added to the principal at stated intervals throughout a given time, the difference between the original principal and the sum due at the end of the time is called *compound interest.* The total amount due is called the *compound amount.* The interest is usually added to the principal at the end of each successive year or half year or quarter year. We say the interest is "*compounded,* or *converted into principal,* annually, semiannually, or quarterly, or m times a year," or that interest is "*payable* annually, or semiannually, or quarterly, or m times a year." The time elapsing between two successive conversions of interest into principal is called the *conversion interval.*

In all theoretical work in compound interest the principal is assumed to be a unit of money, without regard to the particular currency to which the unit belongs. Thus, we speak of the "compound amount on 1" for a given time, and the formulas obtained will apply equally well whether the unit be a dollar, a pound, a mark, or a franc. The compound amount on any principal is then found by multiplying the given principal by the compound amount on a unit principal, since the amount is proportional to the principal employed.

PROBLEM. *To find the compound amount on* 1 *for n years with interest at rate i payable annually.*

At the end of the first year the principal becomes $1+i$, and at the end of the second it is

$$1+i+i(1+i)=(1+i)^2;$$

at the end of the third year it is

$$(1+i)^2+i(1+i)^2=(1+i)^3;$$

and so on. At the end of the nth year it will be the nth term of the geometrical progression

$$1+i, \quad (1+i)^2, \quad (1+i)^3, \quad \cdots,$$

or, denoting the amount by s,

$$s=(1+i)^n. \tag{1}$$

The compound amount of any principal, P, is

$$S=P(1+i)^n. \tag{2}$$

It should be noted that, while the compound amount is proportional to the principal, it is *not* proportional to the rate of interest or to the time, as is simple interest.

ILLUSTRATIVE EXAMPLE. Find the compound amount on $175.50 for 3 years at 5%.

Solution. The compound amount of 1 for 3 years is $(1.05)^3$, or 1.157625. Multiplying this amount by 175.50, we find that the compound amount of $175.50 is $203.16.

PROBLEM. *To find the compound amount on* 1 *compounded m times a year for n years.*

For reasons that will appear later, we use the letter j for the rate instead of i. We assume the rate for the mth part of a year to be the mth part of the rate for a year. The rate for one interval of time will then be $\frac{j}{m}$. The problem is exactly like the first problem, except that the rate per interval of time is $\frac{j}{m}$ and the number of intervals is mn.

The required expression is, then,

$$s=\left(1+\frac{j}{m}\right)^{mn}. \tag{3}$$

Similarly, the compound amount of any principal, P, with interest convertible m times a year at rate j is

$$S=P\left(1+\frac{j}{m}\right)^{mn}. \tag{4}$$

The foregoing definitions and formulas apply only when the time contains an exact number of conversion intervals. Consequently, compound interest can have no meaning when the time does not contain an exact number of conversion intervals. To obviate this difficulty *we assume that formulas* (1) *and* (3), *viz.*

$$s=(1+i)^n \text{ and } s=\left(1+\frac{j}{m}\right)^{mn},$$

hold for all values of n when n is expressed in years.

This assumption is of course equivalent to a definition. According to the definition the compound amount of 1 for 1 year and 6 months at 5%, convertible annually, would be

$$(1.05)^{\frac{3}{2}}=\sqrt{(1.05)^3},$$

and not
$$(1.05)\left(1+\frac{.05}{2}\right),$$

as it is frequently given.

Similarly, the compound amount of 1 for 6 months, at rate .05, convertible annually, would be $(1.05)^{\frac{1}{2}}$, which is *less* than the amount, at simple interest, of the same principal at the same rate and for the same time.

However, in practice the compound amount is usually computed for the integral number of conversion intervals, and the amount of simple interest is then figured on this amount for the fractional part of the interval remaining. For example, the compound amount on 1 for 5 years and 6 months at 6%, payable annually, would be given as

$$(1.06)^5(1.03) = \$1.38.$$

The expression $(1+i)^n$, or its equivalent $\left(1+\frac{j}{m}\right)^{mn}$, is sometimes called an *accumulation factor* (see § 26).

PROBLEM. *To find the rate of interest when the principal, the compound amount, and the time are given.*

(a) If interest is payable annually, the problem is to find i from the formula (2), viz.

$$S = P(1+i)^n.$$

Dividing through by P and extracting the nth root of both sides of the resulting equation, we find

$$1+i = \sqrt[n]{\frac{S}{P}},$$

and, finally,

$$i = \sqrt[n]{\frac{S}{P}} - 1. \tag{5}$$

(b) If interest is convertible m times a year, the relation connecting P, S, n, j, and m is, by (4),

$$S = P\left(1+\frac{j}{m}\right)^{mn}.$$

Solving this equation for j, we find

$$j = m\left(\sqrt[mn]{\frac{S}{P}} - 1\right). \tag{6}$$

The two expressions $\sqrt[n]{\frac{S}{P}}$ and $\sqrt[mn]{\frac{S}{P}}$ are easily found by logarithms when the values of S, P, n, and m are known.

ILLUSTRATIVE EXAMPLE. In 5 years \$1000, placed at interest convertible quarterly, amounts to \$1346.85. What is the rate?

Solution. By formula (6) $$j=4\left(\sqrt[20]{\frac{1346.85}{1000}}-1\right).$$

Since $$\log\sqrt[20]{\frac{1346.85}{1000}}=\frac{1}{20}\times .1293192$$
$$=.0064660,$$

we must have $$\sqrt[20]{\frac{1346.85}{1000}}-1=1.0150-1$$
$$=.0150,$$

and, finally, $$j=.06.$$

PROBLEM. *To find the time when the principal, the amount, and the rate of interest are given.*

Again, it is convenient to divide the solution into two parts, according as interest is convertible one or m times a year.

(a) When interest is payable annually, we have, as before,

$$S=P(1+i)^n,$$

and the problem is reduced to the solution of this exponential equation for n.

Taking logarithms of both sides, we obtain

$$\log S=\log P+n\log(1+i),$$

so that n is easily found to be given by the formula

$$n=\frac{\log S-\log P}{\log(1+i)}. \tag{7}$$

(b) If the interest is convertible m times a year, we find in a similar way, from equation (4),

$$n=\frac{\log S-\log P}{m\log\left(1+\frac{j}{m}\right)}. \tag{8}$$

The problem of finding the conversion interval when the other elements are given cannot be solved by elementary means, since m occurs both as an exponent and as a base.

EXAMPLES

1. In what time would \$236.41 amount to \$421.32, with interest convertible quarterly at 6% nominal?

By formula (8)

$$n = \frac{\log 421.32 - \log 236.41}{4 \log\left(1 + \frac{.06}{4}\right)}$$

$$= \frac{0.2509463}{0.0258640}$$

$$= 9.70.$$

2. In what time will \$2394.62 amount to \$10,000 when placed at compound interest at 5%, convertible semiannually?

7 **23. Nominal and effective rates of interest.** The two problems for which formulas (1) and (3) of the previous section are solutions make necessary a distinction between two kinds of rates of interest when the conversion interval is less than a year, viz. *nominal rate* and *effective rate.* The nominal rate is the rate which would be realized if the interest received at the end of each conversion interval were not productively invested until the end of the year, while the effective rate is the total return on the unit principal for one year. For example, if \$100 were invested at 6%, payable semiannually, with the understanding that the rate per half year be one half the rate per year, the interest for the first half year would be \$3, and the amount available at the beginning of the second half year would be \$103. The interest on \$103 for the second half year would be \$3.09; so that the total amount at the end of the year would be \$106.09. The return on \$1 is therefore .0609, and the effective rate is 6.09%. If the \$3 received at the middle of the year had *not* been productively invested, the total amount at the end of the year would have been \$106, and the rate would have been 6%. This rate is the nominal rate. The nominal rate may also be defined as the product of the rate per conversion interval by the number of conversion intervals in a year. In the example the rate per conversion interval is .03, and the number of conversion intervals in a year is 2, giving .06 as the nominal rate.

PROBLEM. *To express the effective rate in terms of the nominal rate, and vice versa.*

Let i be the effective rate and j the nominal rate convertible m times a year. By formula (3) of § 22 the compound amount of 1 for one year, when the interest is convertible m times a year at rate $\frac{j}{m}$ per conversion interval, is

$$\left(1+\frac{j}{m}\right)^m.$$

The total return on 1 for one year, i.e. the effective rate of interest, is therefore

$$=\left(1+\frac{j}{m}\right)^m-1. \tag{1}$$

We obtain a very useful form of the relation between effective and nominal rates by simply transposing the 1; thus,

$$1+i=\left(1+\frac{j}{m}\right)^m. \tag{2}$$

To obtain j in terms of i we have only to solve (1) or (2) for j. Extracting the mth root of both sides of (2), transposing 1, and, finally, multiplying through by m, we find

$$j=m\{(1+i)^{\frac{1}{m}}-1\}. \tag{3}$$

If we wish to emphasize the fact that j depends upon m as well as i, we write $j_{(m)}$; thus,

$$j_{(m)}=m\{(1+i)^{\frac{1}{m}}-1\}. \tag{3'}$$

The quantity $j_{(m)}$ is an important factor in many computations that will occur later on. Clearly,

$$j_{(1)}=i. \tag{4}$$

The computation required in finding the effective rate when the nominal rate is given, or vice versa, is easily accomplished by means of the binomial expansion, since, according to § 9, the

series will be convergent. If, for example, the nominal rate is .05 convertible quarterly, the effective rate is

$$\begin{aligned} i &= \left(1+\frac{.05}{4}\right)^4 - 1 \\ &= (1.0125)^4 - 1 \\ &= 4\times .0125 + 6\times(.0125)^2 \\ &\qquad + 4\times(.0125)^3 + (.0125)^4 \\ &= .0509+. \end{aligned}$$

The result, true to the fourth decimal place, is obtained by neglecting the last two terms in the binomial expansion.

If the effective rate were .05, and we wished to find the nominal rate when interest is convertible quarterly, we would have

$$\begin{aligned} j &= 4\{(1+.05)^{\frac{1}{4}} - 1\} \\ &= 4\left\{1+\frac{1}{4}\times .05 + \frac{\frac{1}{4}(\frac{1}{4}-1)}{1\cdot 2}(.05)^2 \right. \\ &\qquad \left. + \frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{1\cdot 2\cdot 3}(.05)^3 + \cdots - 1\right\} \\ &= .0491-. \end{aligned}$$

Here again the result obtained by taking four terms of the binomial expansion is accurate to the fourth decimal place.

24. Instantaneous compound interest. Force of interest. From the fundamental formula

$$i = \left(1+\frac{j}{m}\right)^m - 1 \tag{1}$$

it follows that the nominal rate is identical with the effective rate when the interest is converted into principal once a year. Moreover, the effective rate increases as the number of conversion intervals increases, since the positive part of i, viz. $\left(1+\frac{j}{m}\right)^m$, increases. However, it does not increase indefinitely as the number of conversion intervals in a year increases, but approaches a definite and well-known limit. This limit is found by taking the limit of both sides of equation (1). We have, then,

$$\begin{aligned} \lim i &= \lim_{m \doteq \infty}\left[\left(1+\frac{j}{m}\right)^m - 1\right] \\ &= \lim_{m \doteq \infty}\left(1+\frac{j}{m}\right)^m - 1. \end{aligned}$$

But, by § 10, $$\lim_{m \doteq \infty}\left(1+\frac{j}{m}\right)^m = e^j$$

where $$e = 2.71828 +.$$

Denoting $\lim i$ by k for the moment, we have

$$k = e^j - 1. \tag{2}$$

The expression $e^j - 1$, which we have denoted by k, is the *effective rate corresponding to the nominal rate j convertible instantaneously.*

On the other hand, if the effective rate is fixed, and the number of intervals is increased, the corresponding nominal rate as given by the formula

$$j = m\{(1+i)^{\frac{1}{m}} - 1\}$$

is decreased. It is not diminished indefinitely, however, but approaches a limit which is again well known. To find this limit by rough methods, we expand $(1+i)^{\frac{1}{m}}$ by means of the binomial theorem. This method is legitimate, because the resulting series is convergent (see § 9). We have, then,

$$\begin{aligned} j &= m\{(1+i)^{\frac{1}{m}} - 1\} \\ &= m\left\{1 + \frac{1}{m}i + \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)}{1\cdot 2}i^2 + \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1\cdot 2\cdot 3}i^3 + \cdots - 1\right\} \\ &= i + \frac{\left(\frac{1}{m}-1\right)}{1\cdot 2}i^2 + \frac{\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1\cdot 2\cdot 3}i^3 + \cdots. \end{aligned}$$

Assuming that we may take the limit of the right member by taking the sum of the limits of the separate terms, we find

$$\begin{aligned} \lim j &= \lim_{m=\infty}\left[i + \frac{\left(\frac{1}{m}-1\right)}{1\cdot 2}i^2 + \frac{\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1\cdot 2\cdot 3}i^3 + \cdots\right] \\ &= i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \cdots. \end{aligned}$$

The series on the right is exactly the logarithmic series which was discussed in § 11, so that

$$\lim j = \log_e(1+i). \tag{3}$$

It is customary to denote the limit of j by the Greek letter δ (delta), so that

$$\delta = \log_e(1+i). \tag{4}$$

The number δ or its equivalent is called the *force of interest.* The force of interest is the limit of the nominal rate as the number of conversion intervals in a year is indefinitely increased, corresponding to the fixed effective rate i.

To express the effective rate in terms of the force of interest, it is only necessary to write equation (4) in the exponential form. The result is

$$1+i=e^{\delta}, \tag{5}$$

or

$$i=e^{\delta}-1. \tag{6}$$

Formula (6) is identical in form with (2), as indeed it should be, for δ is precisely the limit toward which j approaches as the number of conversion intervals in a year is indefinitely increased, and i is the corresponding effective rate.

The value of i in terms of δ is easily found by means of equation (2) of § 10, which gives

$$i=1+\frac{\delta}{1}+\frac{\delta^2}{1\cdot 2}+\frac{\delta^3}{1\cdot 2\cdot 3}+\cdots-1,$$

or

$$i=\frac{\delta}{1}+\frac{\delta^2}{1\cdot 2}+\frac{\delta^3}{1\cdot 2\cdot 3}+\cdots. \tag{7}$$

For example, if $\delta=.05$,

$$i=\frac{.05}{1}+\frac{(.05)^2}{1\cdot 2}+\frac{(.05)^3}{1\cdot 2\cdot 3}+\cdots$$
$$=.05127+.$$

To find δ when i is given, the logarithmic series

$$\log_e(1+i)=i-\frac{i^2}{2}+\frac{i^3}{3}-\frac{i^4}{4}+\cdots$$

may be used. For example, if $i=.05$,

$$\delta=\log_e(1.05)$$
$$=.05-\frac{(.05)^2}{2}+\frac{(.05)^3}{3}-\frac{(.05)^4}{4}+\cdots$$
$$=.048790+.$$

It is easy to find the value of δ by common logarithms, for, by (5) of § 16,

$$\delta=\log_e(1.05)=\frac{1}{M}\cdot\log_{10}(1.05),$$

where

$$\frac{1}{M}=2.302585+\cdots.$$

Consequently,

$$\delta=2.302585\times .021189+\cdots$$
$$=.048790+\cdots,$$

which agrees with the result already obtained.

Instantaneous compound interest and force of interest have no real existence in practical life, though they are approximated in the business of large concerns which are receiving interest and making loans every day, indeed many times every day. Their value consists chiefly in the fact that by their use many computations are greatly simplified.

If we remember that we have, by formula (6),

$$1 + i = e^{\delta},$$

and that the fundamental relation between nominal and effective rates of interest may take the form

$$1 + i = \left(1 + \frac{j}{m}\right)^m,$$

the formulas for compound interest, viz.

$$s = (1 + i)^n = \left(1 + \frac{j}{m}\right)^{mn}$$

and

$$S = P(1 + i)^n = P\left(1 + \frac{j}{m}\right)^{mn},$$

are expressible in the simple forms

$$s = e^{n\delta} \tag{8}$$

and

$$S = Pe^{n\delta}. \tag{9}$$

These forms justify the name "compound interest law," which was applied to the relation

$$y = a^x,$$

at the close of § 18, for if we put e^{δ} equal to the constant a, (8) becomes

$$s = a^n,$$

where the variables are n and s instead of x and y, as in § 18.

Formulas (8) and (9) are only two of many cases in which a simplification is effected by means of the introduction of the force of interest.

EXAMPLES

1. By means of the exponential series, find the effective rate when the nominal rate is .06 and the interest is compounded momently.

2. Find the force of interest corresponding to the effective rate .06.

3. If the force of interest is .06, find the compound amount of $1259 for 3 years and 6 months.

25. Computation of compound interest. The formulas

$$\begin{aligned} S &= P(1+i)^n \\ &= P\left(1+\frac{j}{m}\right)^{mn} \\ &= Pe^{n\delta} \end{aligned}$$

are admirably adapted to computation by logarithms, but in order to secure accurate results for compound amounts on principals up to $10,000, one would need at least six-place logarithms.

Compound amounts play such an important part in practical affairs that compound-interest tables have been constructed for all the rates in common use and for times up to one hundred years. Such a table gives the amount for unit principal, but the amount for any principal is easily found by multiplication.

Table III gives the amount of 1 at compound interest at rates $1\frac{1}{4}$, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, and 6% for times up to one hundred years. The solutions of the examples that follow will show how to use the table most expeditiously. A little care in noting the number of decimal places that are required will often save a considerable amount of work. For example, if the principal does not exceed \$100, four-place tables will give results accurate to the cent, but if the principal is as large as \$100,000, accurate results cannot be obtained without tables carried out to at least seven places. Eight-place tables would be still better.

If logarithms are not used in performing the multiplication, it is well to begin with the left-hand figure of the multiplier, so that unnecessary figures can be dropped out as the multiplication proceeds. Indeed, it is not necessary to write down multiplicand and multiplier.

Illustrative Example. Required the compound amount of \$236.41 for 10 years with interest payable annually at 5%.

The work may be arranged as follows: Using 5 places, we find from the table that the amount of \$1 for 10 years at 5% is 1.62889. By multiplication we find

Amount of	\$200.00	for 10 years at 5% =	\$325.778
Amount of	30.00	for 10 years at 5% =	48.866
Amount of	6.00	for 10 years at 5% =	9.773
Amount of	.40	for 10 years at 5% =	.651
Amount of	.01	for 10 years at 5% =	.016
Amount of	\$236.41	for 10 years at 5% =	\$385.084

By utilizing the properties of a power of a base the range of the table may be considerably extended. If, for example, we wished to find the amount of \$236.41 for 75 years at 5%, with only a fifty-year table at hand, we could write

$$\begin{aligned} s &= \$236.41 \times (1.05)^{50} \times (1.05)^{25} \\ &= \$236.41 \times 11.46740 \times 3.38635. \end{aligned}$$

The value of this product may be found by means of a table of six-place logarithms.

If interest is convertible oftener than once a year, we can use the table for the rate $\frac{j}{m}$ and for mn years. For example, to find the amount of 1 for 10 years at 5%, convertible semiannually, we have to look up the amount of 1 for 20 years at $2\frac{1}{2}$%, convertible annually, since, by the formula, we have

$$s=\left(1+\frac{.05}{2}\right)^{10\times 2}=(1.025)^{20}.$$

If the time does not contain the conversion interval an exact number of times, it will ordinarily be necessary to use logarithms or straight-out multiplication to reach the result. If, for example, we wish to find the amount of \$250 for 5 years and 6 months at 5% when interest is payable annually, we must find the value of the product

$$250\times(1.05)^{\frac{11}{2}}.$$

If a table of logarithms is not at hand, the value of $(1.05)^{\frac{11}{2}}$ may be found by the binomial theorem, or we may take the value of $(1.05)^5$ from the table and find the value of $(1.05)^{\frac{1}{2}}$, either by the binomial formula or by simple root extraction.

MISCELLANEOUS EXAMPLES

1. Find the compound amounts in the following examples:

(a) \$239.54 at 5%, convertible annually, for 6 years.
(b) \$250 at 5%, convertible annually, for 3 years and 6 months.
(c) \$300 at 5%, convertible annually, for 120 years.
(d) \$1000 at 6%, convertible quarterly, for 20 years.

2. How long will it take for a sum to double itself at 5%, convertible annually?

3. At a certain university which had 4000 students in 1910 the attendance has been increasing at the rate of 10% each year over the previous year's attendance. If the rate of increase should be kept up, what would be the attendance in 1920?

4. A merchant starting out with a capital of \$5000 finds that after 8 years his capital is \$10,000. What has been the annual rate of increase if the rate of increase is supposed to have been uniform through the 8-year period?

5. A piece of timber increases in value each year by 4% of the previous year's value for 10 years. If its value at the beginning of the period was \$40,000, what would be the value at the end of the 10 years?

6. At the beginning of the year 1907 the visible coal supply of the United States was estimated to be 3,076,204,000,000 tons, and in that year the consumption was 480,363,424 tons, which was 7.36% in excess of the consumption in 1906. Assuming that consumption will go on increasing at the same rate, how long will it be before the supply is exhausted? — President Van Hise in lectures on conservation.

7. What would be the formula for compound interest when the interest is payable in advance, as in the case of small loans made from banks?

8. A rule in common use for finding the time in which a sum of money will double itself at compound interest is, "Divide .69 by the rate of interest and add one third of a year." Prove that the rule is approximately correct. (The Napierian and not the common logarithm of 2 must be used.)

9. Find a rough rule for the time in which any sum of money will quadruple itself at compound interest.

10. A banker charges 7% payable in advance on loans made for 90 days. What is the effective rate?

11. A man leaves to a university an estate valued at \$2,000,000, on condition that one half the annual income is to be added to the principal until the whole amount reaches \$20,000,000, after which one fourth of the income is to be added to the principal until the whole amount reaches \$30,000,000. If the funds can be made to yield 5%, when will the value of the estate reach \$20,000,000 and when will it reach \$30,000,000? Solve the problem on the basis of 4%.

12. Construct the graph showing the compound amount of 1 at 4%, payable annually, as the time varies.

13. Construct the graph for the amount of 1 at simple interest at 4% as the time varies, and compare it with the graph in the preceding problem. For what times do the two graphs show the two amounts to be equal?

26. Discount. The word *discount* has a variety of meanings in the world of business. To the merchant buying a stock of goods it means a reduction in his bill for the payment of cash; to the same merchant selling his goods it means a reduction from the marked price of an article to secure prompt sale; to the banker it usually means simple interest payable in advance.

The problems that arise in such cases are usually problems in percentage or in simple interest, and for that reason they do not require discussion here.

Discount, as we shall use the term, is a consideration for the payment of a sum of money before it is due. It is the difference between the value of a sum of money payable at some future time, with or without interest, at the time when it is due, and its value at some earlier time, usually the present. We sometimes say of a sum of money that it is *accumulated* to a certain date, meaning thereby that it is put at interest until the date in question; on the other hand, we say that a sum of money is *discounted* to a certain date, meaning that we seek to find the value of the sum at the date in question as compared with its value given at some assigned later date.

The value *now* of a sum due at some future date, with or without interest, is called the *present worth*, or *present value*. The present value of a sum due at some future date may also be defined as that sum which, put at compound interest now, would amount to the sum in question on the given future date. The processes of accumulation and discounting are then exactly the reverse of each other. A clear recognition of this fact will be of the greatest aid in understanding future sections.

PROBLEM. *To find the present value of* 1 *due in n years without interest.*

Let v_n be the required present value, and let i be the rate of interest that might be obtained if the unit principal were in hand to put at interest. By the definition of present value, we have

$$v_n(1+i)^n = 1;$$

whence

$$v_n = \frac{1}{(1+i)^n}. \tag{1}$$

If the principal be not 1, it may be denoted, as in the interest formulas, by P. Let V_n denote the present value of P due in n years; then

$$V_n = P\frac{1}{(1+i)^n} = P(1+i)^{-n}. \tag{2}$$

Illustrative Example. Find the present value of \$2365.29 due 3 years hence, if money could be invested at 5%.

Solution. By formula (2)

$$V_3 = \$2365.29\,(1.05)^{-3}.$$

The value of the product on the right may be obtained by means of a table of six-place logarithms or by means of a table of present values of \$1, from which the value of $(1.05)^{-3}$ is taken directly. If neither the table of logarithms nor the table of present values is at hand, the value of $(1.05)^{-3}$ may be obtained by the binomial expansion. Using the last-named method, we have

$$(1.05)^{-3} = 1 + (-3)(.05) + \frac{(-3)(-3-1)}{1\cdot 2}(.05)^2$$

$$+ \frac{(-3)(-3-1)(-3-2)}{1\cdot 2\cdot 3}(.05)^3 + -.$$

Six terms of this series give the result

$$(1.05)^{-3} = .863837 + \cdots.$$

Consequently,

$$V_3 = \$2365.29 \times .863837$$
$$= \$2043.22.$$

Problem. *To find the present value of* 1 *due n years hence with interest.*

Let k be the rate of interest borne by the unit principal, i the current rate, and v_n' the present value. In this problem the sum due in n years is not 1, but the amount of 1 for n years at rate k, viz. $(1+k)^n$. Consequently,

$$v_n'(1+i)^n = (1+k)^n$$

and

$$v_n' = \frac{(1+k)^n}{(1+i)^n}. \tag{3}$$

If the principal is P and the corresponding present value is V_n',

$$V_n' = P\frac{(1+k)^n}{(1+i)^n}. \tag{4}$$

Formulas (1)–(4) may be expressed in terms of the nominal rates j and k' by means of the fundamental relations

$$1+i = \left(1+\frac{j}{m}\right)^m$$

and

$$1+k = \left(1+\frac{k'}{m'}\right)^{m'}.$$

ILLUSTRATIVE EXAMPLE. Find the present value of \$379.25 with interest payable annually at 4% and due 3 years hence when money is worth 5% nominal, payable semiannually.

Solution. By formula (4),

$$V'_3 = \$379.25\frac{(1.04)^3}{(1.025)^6},$$

where the nominal rate has been introduced by means of the relation

$$\left(1+\frac{.05}{2}\right)^2 = (1+i).$$

The result is found by logarithms to be \$367.86.

The expression $\frac{1}{1+i}$ occurs so often and is of such importance in what follows that it is customary to denote it by a single letter, v. We have, then, as an equation defining v,

$$v = \frac{1}{1+i} = (1+i)^{-1}. \tag{5}$$

Replacing $(1+i)^{-1}$ by v, the formulas (2) and (4) become

$$V_n = Pv^n \tag{6}$$

and

$$V'_n = P(1+k)^n v^n. \tag{7}$$

The expression

$$v^n = (1+i)^{-n}$$

is sometimes called the *discounting factor*, just as $(1+i)^n$ has been called an accumulation factor (§ 22). A given sum is discounted by multiplying it by v^n.

Formulas (3) and (4), or the equivalent formulas (6) and (7), are sufficient for the solution of all ordinary problems requiring the determination of present values or discounts. In practice the work is usually done by means of a table of values of $v^n = (1+i)^{-n}$. Table IV gives the present values of 1 at rates $1\frac{1}{4}$, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, and 6% for times up to one hundred years.

27. The rate of discount. The measure of the discount on unit principal for unit time is called the *rate of discount.* Since present value has been defined as the principal diminished by the discount, we have, as a defining equation,

$$v = 1 - d, \tag{1}$$

where d denotes the rate of discount.

From equation (1) we may easily find the relation between the rate of discount and the rate of interest, for, replacing v by its value as given by (1) (§ 26),

$$\frac{1}{1+i}=1-d.$$

Solving this equation for d,

$$d=\frac{i}{1+i}=iv. \tag{2}$$

Solving the same equation for i,

$$i=\frac{d}{1-d}. \tag{3}$$

Formulas (2) and (3) furnish the means for the direct computation of i when d is known, and vice versa. For example, if $i=.06$, then, by simple division,

$$d=\frac{.06}{1.06}=.05660+\cdots.$$

In many cases, however, it is even easier to express the one quantity in terms of the other by means of a power series, and to use the power series as a means of computation. By division we obtain from (2) the formula

$$d=i-i^2+i^3-i^4+\cdots. \tag{4}$$

Similarly, from (3),

$$i=d+d^2+d^3+\cdots. \tag{5}$$

Illustrative Example. What is the rate of discount when the rate of interest is .05?

Solution. By (4),

$$\begin{aligned} d &= .06-(.06)^2+(.06)^3-(.06)^4+\cdots \\ &= .05660+\cdots. \end{aligned}$$

Additional terms would not change this result unless more decimal places were used.

The computation of i by means of (5) is even simpler, since all the terms of the series are positive.

Note. The so-called *true discount* is defined as the difference between the principal and the sum which, put at *simple* interest, would amount to the principal when the latter becomes due.

Denoting the present value of 1 discounted by true discount by v'_n, the definition gives

$$v'_n(1 + ni) = 1;$$

so that

$$v'_n = \frac{1}{1 + ni}. \tag{6}$$

Similarly, if V'_n denote the present value of P,

$$V'_n = \frac{P}{1 + ni}. \tag{7}$$

True discount is rarely used in practice. The fact that it cannot be used in problems where it is assumed that interest is paid promptly is brought out by means of Example 8, below. In the second part of this example the payments must be discounted by compound and not by true discount, in order to make the purchase price of the farm $5000, as it should be.

EXAMPLES

1. Find the present value of $1239 due in 2 years without interest when money is worth 5%.

2. Find the present value of a debt of $1239 with interest payable annually at 6% for 2 years, when money is worth 5%.

3. A father wishes to set aside, at the birth of his son, a sum that will accumulate to $5000 by the time the son reaches his majority. If money is worth 5%, what is the sum required?

4. Find the present value of $1225 due 3 years and 6 months hence without interest when money is worth 5%.

5. What is the rate of discount when the rate of interest is .05?

6. What is the rate of interest corresponding to the rate of discount .05?

7. A wholesale merchant sells his goods on 90 days' time or 3% off for cash. What rate of interest is equivalent to this discount? In other words, what would be the highest rate the buyer could afford to pay to borrow money to pay cash?

8. One man buys a farm for $5000, agreeing to pay $1000 cash and $1000 with interest at 6% on all sums remaining due at the end of each year, until the whole amount is paid. How much does he pay each year?

Another man buys a farm, agreeing to pay $1000 cash, $1240 at the end of the first year, $1180 at the end of the second, $1120 at the end of the third, and $1060 at the end of the fourth. If money is worth 6%, what is the cash price of the second man's farm?

Explain the significance of these two problems in the light of the note at the end of § 27.

9. Find the rate of discount when the nominal rate of interest is .06, convertible semiannually.

10. A restaurant keeper sells a meal ticket good for four dollars' worth of meals to be served in a month for \$3.75, payable in advance. Assuming that the \$4.00 could be paid equitably at the middle of a month of 30 days, what rate of interest is equivalent to the discount?

11. Construct the graph of the function $(1.04)^{-n}$ with the time as the variable. What is the significance of that part of the graph that lies to the left of the axis of ordinates?

28. The equation of value. It is frequently necessary to compare the value of one set of sums due at various times with another set due at other times. In order that such comparison may be possible, we must observe the following **fundamental principle**: *To compare several sums due at various times, all the sums must be accumulated or discounted to the same date.*

In case one set of obligations is equivalent to another set, the relation expressing the fact is called the *equation of value.* The formation of the equation of value can be best illustrated by an example. Suppose a man owing \$1000 due in 18 months, and \$500 due in 2 years, wishes to discharge the obligations by two equal payments, one made in 6 months and the other in 1 year.

Let the value of each of the two equal payments be denoted by X. We have then to consider four sums: X due in 6 months, X due in 1 year, \$1000 due in 18 months, and \$500 due in 2 years. If all these sums be discounted to the present, their present values will be $Xv^{\frac{1}{2}}$, Xv, $1000\,v^{\frac{3}{2}}$, and $500\,v^2$, respectively. By hypothesis, the first two obligations are exactly equivalent to the second two. We have, then, as the equation of value,

$$Xv^{\frac{1}{2}} + Xv = 1000\,v^{\frac{3}{2}} + 500\,v^2. \tag{1}$$

This equation, evidently, suffices to determine the value of the unknown number X.

The time to which all the sums are accumulated, or discounted, is wholly immaterial. For example, if, in the present problem, we accumulate the various sums to the last date, the accumulation periods will be $1\frac{1}{2}$, 1, $\frac{1}{2}$, and 0 years, respectively. The equation of value will then be

$$X(1+i)^{\frac{3}{2}} + X(1+i) = 1000\,(1+i)^{\frac{1}{2}} + 500\,(1+i)^0.$$

Dividing this equation through by $(1+i)^2$, we obtain

$$X(1+i)^{-\frac{1}{2}}+X(1+i)^{-1}=1000\,(1+i)^{-\frac{3}{2}}+500\,(1+i)^{-2},$$

which is identical with equation (1). Again, if we multiply equation (1) by v^{100}, we obtain an equation of value in which all sums have been discounted to a date 100 years ago. The equation would be unchanged, however, except in form.

The equation of value in its most general form would be an equation expressing the fact that a series of sums $X_1, X_2, \cdots, X_r$, due in $n_1, n_2, \cdots, n_r$ years, respectively, is equivalent to another set $Y_1, Y_2, \cdots, Y_s$, due in $m_1, m_2, \cdots, m_s$ years. Discounting all sums to the present, we have for the equation of value

$$X_1v^{n_1}+X_2v^{n_2}+\cdots+X_rv^{n_r}=Y_1v^{m_1}+Y_2v^{m_2}+\cdots+Y_sv^{m_s}. \quad (2)$$

The principle involved in the equation of value should be thoroughly mastered, for it forms the basis for the solution of a very considerable number of important problems.

29. Equation of payments. The term *equation of payments*, or *equation of accounts*, is used to denote a rule for determining the time at which several debts, due at different times, can be equitably discharged by the payment of a single sum equal in amount to the sum of all the debts. The rule is found by solving an equation of value. The time thus found is called the *equated time.*

PROBLEM. *To determine the equated time for the payment of several debts due at various times.*

Let $s_1, s_2, s_3, \cdots, s_r$, denote the sums due at later dates, and $n_1, n_2, n_3, \cdots, n_r$, the times to elapse before they fall due. If n denote the equated time, and all sums be discounted to the present, the equation of value will be

$$(s_1+s_2+\cdots+s_r)\,v^n=s_1v^{n_1}+s_2v^{n_2}+s_3v^{n_3}+\cdots+s_rv^{n_r}. \quad (1)$$

Solving this equation for v^n, we find

$$v^n=\frac{s_1v^{n_1}+s_2v^{n_2}+\cdots+s_rv^{n_r}}{s_1+s_2+\cdots+s_r}.$$

Solving this exponential equation for n, we find

$$n = \frac{\log\left[s_1 v^{n_1} + s_2 v^{n_2} + \cdots + s_r v^{n_r}\right] - \log\left[s_1 + s_2 + \cdots + s_r\right]}{\log v}. \quad (2)$$

If we replace v by its value $(1+i)^{-1}$, equation (2) may be written in the form

$$n = \frac{\log[s_1 + s_2 + \cdots + s_r] - \log[s_1 v^{n_1} + s_2 v^{n_2} + \cdots + s_r v^{n_r}]}{-\log(1+i)^{-1}}. \quad (3)$$

The equated time will be computed by means of formula (2) or formula (3).

The *ordinary rule for the equation of payments* is obtained by substituting in equation (1) approximate values for $v^n, v^{n_1}, v^{n_2}, \cdots, v^{n_r}$. It is stated as follows:

Multiply each sum by the time to elapse before it becomes due, add the products, and divide the sum by the sum of all the amounts to fall due.

To obtain the formula which gives the rule, consider the expansions

$$v^n = (1+i)^{-n} = 1 - ni + \frac{(-n)(-n-1)}{1\cdot 2} i^2 + \cdots,$$

$$v^{n_1} = (1+i)^{-n_1} = 1 - n_1 i + \frac{(-n_1)(-n_1-1)}{1\cdot 2} i^2 + \cdots,$$

$$\vdots$$

$$v^{n_r} = (1+i)^{-n_r} = 1 - n_r i + \frac{(-n_r)(-n_r-1)}{1\cdot 2} i^2 + \cdots.$$

Dropping powers of i higher than the first, and substituting for $v^n, v^{n_1}, v^{n_2}, \cdots, v^{n_r}$, the approximate values,

$$1 - ni, \quad 1 - n_1 i, \quad 1 - n_2 i, \quad \cdots, \quad 1 - n_r i,$$

in equation (1), we obtain the equation

$$(s_1 + s_2 + s_3 + \cdots + s_r)(1 - ni) = s_1(1 - n_1 i) + s_2(1 - n_2 i) + \cdots + s_r(1 - n_r i).$$

Solving this equation for n, we find

$$n = \frac{n_1 s_1 + n_2 s_2 + \cdots + n_r s_r}{s_1 + s_2 + \cdots + s_r}. \tag{4}$$

Equation (4) gives the ordinary rule that is used in practical work. The ordinary rule favors the debtor by slightly increasing the equated time, though it is reasonably accurate where the periods of time involved are short. (See Todhunter, "Institute of Actuaries' Text-Book," Chap. II, Art. 8.)

EXAMPLES

1. What sum due 9 months hence without interest will be the equivalent of three debts, of $500, $400, and $700, due 6, 8, and 12 months hence, respectively, when money is worth 5%?

2. A man owes the following sums: $500 due in 6 months without interest, $700 due in 1 year without interest, $600 due in 2 years with interest at 5%, payable annually, and $300 due in 1 year with interest at 4½%, payable annually. He wishes to arrange to discharge his indebtedness by three equal annual payments, the first to be paid 1 year hence. If money is worth 5%, what will be the annual payment?

3. A man is offered $2000 cash and $1000 at the end of each year for 3 years for a house and lot, or $1250 cash and $1250 at the end of each year for 3 years. Which is the better offer if money is worth 5%?

4. Find the difference between the equated times by the ordinary rule and by the exact rule for the following sums, due without interest, when money is worth 5%: $600 due in 1 year, $700 due in 3 years, $400 due in 2 years, and $1000 due now.

CHAPTER VI

ANNUITIES

30. Definitions and notation. An *annuity* is a series of payments, usually equal in amount, made at equal intervals of time, called *intervals of payment.* Unless the contrary is specifically stated, the first payment is made at the *end* of the first interval. If the first payment is made at the beginning instead of at the end of the first interval, the annuity is called an *annuity due.*

An *annuity certain* is one for which the payments begin and end at fixed dates. The time to elapse between the beginning of the first interval of payment and the end of the last is called the *term* of the annuity certain.

If the date either of the first or of the last payment depends upon some event, the time of whose occurrence cannot be foretold, the annuity is called a *contingent annuity.* An annuity whose payments begin or end with the death of an individual would be a contingent annuity.

A *deferred annuity* is one for which a certain specified number of intervals must elapse before the first payment is made. A deferred annuity becomes an ordinary annuity after the lapse of the specified number of intervals. To be exact, if the annuity is deferred m intervals, the first payment is made at the end of the $(m+1)$ st interval.

A *perpetuity* is an annuity whose payments are supposed to continue forever.

When the payments are equal in amount, the sum of the payments made in one year is called the *annual rent.* If the annual rent be R, we speak of *an annuity of R per annum.*

If the payments of an annuity are allowed to accumulate to the end of the term during which they are made, the annuity is said to be *forborne*, and the total sum due at the end of this term is called the *amount* of the annuity.

The *present value*, or the cash equivalent, of an annuity is the sum of the present values of all the payments of the annuity.

One has only to cite a few examples to see the wide range of application of the theory of annuities. The rental of a house, the interest payments on a mortgage note, the annual premiums on a life-insurance policy, the stated return from an interest-bearing bond, a pension, are all examples of annuities. In fact, any process involving the stated payment of a sum of money may be looked upon as giving rise to an annuity.

The following notation is standard among English-speaking writers on the subject of annuities:

$s_{\overline{n}|}$ denotes the amount of an annuity of 1 per annum, payable annually for n years.

$s_{\overline{n}|}^{(p)}$ denotes the amount of an annuity of 1 per annum, payable in p installments at equal intervals throughout the year for n years.

$a_{\overline{n}|}$ denotes the present value of 1 per annum, payable annually for n years.

$a_{\overline{n}|}^{(p)}$ denotes the present value of 1 per annum, payable p times a year for n years.

$_{m|}a_{\overline{n}|}^{(p)}$ denotes the present value of a deferred annuity of 1 per annum, payable p times a year for n years, the first payment to be made after the lapse of $m + \dfrac{1}{p}$ years.

If the annuity is due, the roman "fullface" **a** is used for the present value. Thus, $\mathbf{a}_{\overline{n}|}$ and $\mathbf{a}_{\overline{n}|}^{(p)}$ are used to denote the present values of the annuities due, the first of 1 per annum, payable annually for n years, and the second of 1 per annum, payable p times a year for n years.

When interest is convertible oftener than once a year, the rate of interest may be given as either effective or nominal. All formulas involving the rate of interest may be expressed in the one or the other of these at will by means of the fundamental relation

$$(1+i) = \left(1+\frac{j}{m}\right)^m.$$

All theoretical computations are made for an annuity of 1 per annum.

31. The amount of an annuity.

PROBLEM. *To find the amount of a forborne annuity of* 1 *per annum, payable annually for n years.*

We will first find the required formula in terms of the effective rate i.

The first payment made one year from the beginning of the term of the annuity will be accumulated for $n-1$ years and will amount to $(1+i)^{n-1}$. Likewise, the second will amount to $(1+i)^{n-2}$, the third to $(1+i)^{n-3}$, and so on, while the last term will be cash. The series of amounts will be

$$(1+i)^{n-1}, \quad (1+i)^{n-2}, \cdots, (1+i), \quad 1.$$

Denoting the total amount by $s_{\overline{n}|}$, according to the notation established in § 30, and writing the terms in reverse order, we have

$$s_{\overline{n}|}=1+(1+i)+(1+i)^2+\cdots+(1+i)^{n-2}+(1+i)^{n-1}.$$

The expression on the right is a geometrical progression of n terms with ratio $1+i$, and the sum is found by the formula

$$s=\frac{rl-a}{r-1}. \quad \text{(See formula } (D'), \text{ § 2.)}$$

Substituting for r, l, and a their values $1+i$, $(1+i)^{n-1}$, and 1, we find, as the required formula,

$$\mathbf{s_{\overline{n}|}=\frac{(1+i)^n-1}{i}}. \qquad (1)$$

If interest is convertible m times a year at nominal rate j, we may replace i and $(1+i)^n$ by their values as obtained in the fundamental relations (1) and (2) of § 23. The result will be

$$s_{\overline{n}|}=\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{m}-1} = \frac{s_{\overline{mn}|}^{\text{at } \frac{j}{m}}}{s_{\overline{m}|}^{\text{at } j/m}} \qquad (2)$$

Formula (2) gives the amount of the annuity in terms of the nominal rate j.

PROBLEM. *To find the amount of an annuity of* 1 *per annum, payable* p *times a year for* n *years.*

Suppose the rate of interest is the effective rate i. The time to elapse between the making of the first payment and the end of the whole transaction will be $n - \frac{1}{p}$ years; and, similarly, the times for the second and succeeding payments will be $n - \frac{2}{p}$ years, $n - \frac{3}{p}$ years, and so on. The last payment will be cash. The amounts of the various payments, beginning with the first, will be

$$\frac{1}{p}(1+i)^{n-\frac{1}{p}},\quad \frac{1}{p}(1+i)^{n-\frac{2}{p}},\quad \cdots,\quad \frac{1}{p}(1+i)^{\frac{1}{p}},\quad \frac{1}{p}.$$

Writing these amounts in reverse order and denoting their sum by $s_{\overline{n}|}^{(p)}$, according to § 30, we have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} + \frac{1}{p}(1+i)^{\frac{1}{p}} + \frac{1}{p}(1+i)^{\frac{2}{p}} + \cdots + \frac{1}{p}(1+i)^{n-\frac{1}{p}}.$$

This is again a geometrical progression with first term $\frac{1}{p}$, ratio $(1+i)^{\frac{1}{p}}$, and last term $\frac{1}{p}(1+i)^{n-\frac{1}{p}}$. By formula (D') of § 2 the sum will be

$$s_{\overline{n}|}^{(p)} = \frac{1}{p}\cdot\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1}. \tag{3}$$

If we desire the formula for the amount in terms of a nominal rate j, convertible m times a year, we have only to replace $1+i$ by its value in terms of j, viz. by $\left(1+\frac{j}{m}\right)^m$, and we obtain the formula

$$s_{\overline{n}|}^{(p)} = \frac{1}{p}\,\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1}. \tag{4}$$

Finally, if the interval between the successive payments of the annuity coincides with the conversion interval, and, consequently, $m = p$, formula (4) reduces to

$$s_{\overline{n}|}^{(p)} = \frac{1}{p}\,\frac{\left(1+\dfrac{j}{p}\right)^{np}-1}{\dfrac{j}{p}} = \frac{1}{p}\,s_{\overline{np}|} \text{ at rate } \frac{j}{p}. \tag{5}$$

The special case for which the solution is given by formula (5) is important in later work.

If the annual rent of an annuity is R, the amount K will be

$$K = Rs_{\overline{n}|} = R\,\frac{(1+i)^n-1}{i}, \tag{6}$$

or

$$K = Rs_{\overline{n}|}^{(p)} = R\,\frac{(1+i)^n-1}{p\left[(1+i)^{\frac{1}{p}}-1\right]}, \tag{6'}$$

as the case may be. The corresponding formulas when the rate is nominal, payable m times a year, will be found by replacing $1+i$ by $\left(1+\dfrac{j}{m}\right)^m$.

EXAMPLES

1. A man puts $100 into a savings bank at the end of every year for 5 years. If the bank pays 4%, payable annually, what will his savings amount to at the end of the 5 years?

Solution. The amount will be found by substituting the values $n = 5$ and $i = 4$ in formula (1), and then multiplying by 100. The result is

$$\$100 \cdot \frac{(1.04)^5-1}{.04} = \$541.63.$$

2. Suppose, as is usually the case, the savings bank pays interest twice a year at nominal rate .04; to what would the savings amount in the previous example? In this case $j = 4$, $m = 2$, $p = 2$, and formula (4) or (5) becomes

$$\begin{aligned}\$100\,s_{\overline{5}|}^{(2)} &= \$100 \cdot \tfrac{1}{2} \cdot \frac{(1.02)^{10}-1}{.02}\\ &= \$547.49.\end{aligned}$$

3. Find the amount of an annuity of $500 a year for 10 years accumulated at 4%.

4. A man saves $400 a year for 10 years; what will his savings amount to in 10 years if they are accumulated at 4%?

32. The present value of an annuity.

PROBLEM. *To find the present value of an annuity of* 1 *per annum, payable annually for n years.*

Let it be required to find the present value in terms of the effective rate i. The several annual payments will be, in effect, a series of sums due in 1, 2, 3, $\cdots$, n years. Their present values will then be, by (5), § 26,

$$(1+i)^{-1}=v, \quad (1+i)^{-2}=v^2, \quad \cdots, \quad (1+i)^{-n}=v^n.$$

Denoting the sum of the present values by $a_{\overline{n}|}$, we have

$$a_{\overline{n}|}=v+v^2+v^3+\cdots+v^n.$$

The right member is a geometrical progression with first term v, last term v^n, and ratio v. The sum is therefore

$$a_{\overline{n}|}=\frac{v^{n+1}-v}{v-1}=\frac{v-v^{n+1}}{1-v}.$$

If we remember that v is defined by the equation

$$v=(1+i)^{-1},$$

the expression on the right reduces easily by dividing numerator and denominator by v, so that

$$\boldsymbol{a_{\overline{n}|}=\frac{1-v^n}{i}=\frac{1-(1+i)^{-n}}{i}}. \tag{1}$$

If the nominal rate j is given, the formula (1) may be written

$$a_{\overline{n}|}=\frac{1-v^n}{\left(1+\frac{j}{m}\right)^m-1}=\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^m-1}. \tag{2}$$

PROBLEM. *To find the present value of an annuity of* 1 *per annum, payable p times a year for n years.*

The stated payment will be $\frac{1}{p}$, and the times to elapse before the several payments are to be made will be, in years and fractions of a year,

$$\frac{1}{p}, \quad \frac{2}{p}, \quad \frac{3}{p}, \quad \cdots, \quad n-\frac{1}{p}, \quad n.$$

The present values in terms of the effective rate i will be

$$\frac{1}{p}(1+i)^{-\frac{1}{p}}=\frac{1}{p}v^{\frac{1}{p}},\ \frac{1}{p}(1+i)^{-\frac{2}{p}}=\frac{1}{p}v^{\frac{2}{p}},\ \ldots,\ \frac{1}{p}(1+i)^{-n}=\frac{1}{p}v^{n}.$$

Denoting the sum of these present values by $a_{\overline{n}|}^{(p)}$, we have

$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\left(v^{\frac{1}{p}}+v^{\frac{2}{p}}+\cdots+v^{n-\frac{1}{p}}+v^{n}\right).$$

The right member is a geometrical progression of np terms, with first term $\frac{1}{p}v^{\frac{1}{p}}$, last term $\frac{1}{p}v^{n}$, and ratio $v^{\frac{1}{p}}$. Consequently, the sum is

$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\,\frac{v^{n+\frac{1}{p}}-v^{\frac{1}{p}}}{v^{\frac{1}{p}}-1}.$$

Dividing numerator and denominator by $v^{\frac{1}{p}}$, and noting that

$$v=(1+i)^{-1},$$

we find
$$a_{\overline{n}|}^{(p)}=\frac{1-v^{n}}{p\left[(1+i)^{\frac{1}{p}}-1\right]}=\frac{1-(1+i)^{-n}}{p\left[(1+i)^{\frac{1}{p}}-1\right]}. \tag{3}$$

If the nominal rate j, convertible m times a year, is given, we must express $1+i$ in terms of $1+\frac{j}{m}$ by the fundamental relation

$$1+i=\left(1+\frac{j}{m}\right)^{m};$$

so that
$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\,\frac{1-v^{n}}{\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1}=\frac{1}{p}\,\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1}. \tag{4}$$

Again, if the conversion interval for interest coincides with the payment interval for the annuity, $m=p$, and (4) becomes

$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\,\frac{1-v^{n}}{\frac{j}{p}}=\frac{1}{p}\,\frac{1-\left(1+\frac{j}{p}\right)^{-np}}{\frac{j}{p}}=\frac{1}{p}a_{\overline{np}|}\text{ at rate }\frac{j}{p}. \tag{5}$$

The case covered by formula (5) occurs frequently in practice.

If the annual rent of an annuity is R, then R will be a factor of every term in the progression whose sum is given by formula (1) or formula (3). Hence, R will be a factor of the sum, and we have as formulas expressing the present value A of an annuity whose annual rent is R,

$$A = Ra_{\overline{n}|} = R\frac{1-v^n}{i}, \tag{6}$$

and

$$A = Ra_{\overline{n}|}^{(p)} = \frac{R}{p}\frac{1-v^n}{(1+i)^{\frac{1}{p}}-1}. \tag{6'}$$

The formulas are similar when expressed in terms of the nominal rate j, payable m times a year.

EXAMPLES

1. A man buys a house, agreeing to pay \$1000 cash and \$1000 at the end of each year for five years. What would be the cash price of the house (*a*) if money is worth 6% effective? (*b*) if money is worth 6% nominal, payable semiannually?

Solution. The first \$1000 constitutes a cash payment whose present value is \$1000, and the other five payments constitute an annuity of \$1000 per annum. For the first case the present value of the annuity, as given by formula (1), is

$$\$1000\frac{1-(1.06)^{-5}}{.06} = \$4212.36.$$

The cash price would then be \$5212.36.

For the second case the present value of the annuity, as given by formula (2), is

$$\$1000\frac{1-(1.03)^{-10}}{(1.03)^2-1} = \$4202.07,$$

and the cash price would be \$5202.07.

2. Find the present value of an annuity of \$395.47 per annum, payable twice a year, for 10 years, with interest at 6% nominal, convertible twice a year.

33. Annuities due and deferred annuities. An annuity due has been defined in § 30 as one whose first payment is made at the beginning instead of the end of the first interval.

The present value of an annuity due may be expressed easily in terms of the present value of an ordinary annuity, for the

first payment of an annuity due is cash, and the remainder of the payments constitute an ordinary annuity with one interval less. We have therefore

$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{1}$$

Similarly, for an annuity due, payable p times a year,

$$\mathbf{a}^{(p)}_{\overline{n}|} = \frac{1}{p} + a^{(p)}_{\overline{n-\frac{1}{p}}|}. \tag{2}$$

The amount of an annuity due is used by continental writers in place of the amount of an ordinary annuity. Denoting the amount of an annuity due by $\mathbf{s}_{\overline{n}|}$, and remembering that the first payment is at interest for n years, the second for $n-1$ years, and so on, we have

$$\begin{aligned}\mathbf{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)\\ &= (1+i)\left[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1\right];\end{aligned}$$

so that

$$\mathbf{s}_{\overline{n}|} = (1+i)\, s_{\overline{n}|} = s_{\overline{n+1}|} - 1. \tag{3}$$

Similarly, for an annuity payable p times a year we have

$$\mathbf{s}^{(p)}_{\overline{n}|} = (1+i)^{\frac{1}{p}}\, s^{(p)}_{\overline{n}|}.* \tag{4}$$

A deferred annuity has been defined as one whose term does not begin until the expiration of a fixed number of years. If an annuity payable for n years be deferred for m years, $m+n$ years will elapse before the last payment is made. The annuity may be looked upon as an annuity for $m+n$ years, for which the payments for the first m years are withheld. The present value of the annuity for n years deferred m years is, then the present value of the annuity for $m+n$ years diminished by the present value of the annuity for m years under the same conditions. In symbols,

$$m|a^{(p)}_{\overline{n}|} = a^{(p)}_{\overline{m+n}|} - a^{(p)}_{\overline{m}|}. \tag{5}$$

Formula (5) holds for all values of p, whatever the conversion interval for the interest may be. The amount of a deferred

*Ordinarily the last payment of an annuity is looked upon as closing the transaction. On the date of the last payment of an annuity due the amount would be $s_{\overline{n}|}$ or $s^{(p)}_{\overline{n}|}$, according as the payments are made annually or p times a year.

annuity at the end of its term would be the same as the amount of an ordinary annuity having the same term; i.e.

$$m|s_{\overline{n}|}^{(p)} = s_{\overline{n}|}^{(p)}$$

for all values of m.

ILLUSTRATIVE EXAMPLE. Use the formulas of the present section to solve Example 1 at the end of the previous section.

Solution. If money is worth 6% effective, the present value of the six payments, of which the first is cash, is

$$\begin{aligned}\$1000\, \text{a}_{\overline{6}|} &= \$1000\,(1 + a_{\overline{5}|})\\ &= \$1000\left(1 + \frac{1-(1.06)^{-5}}{.06}\right)\\ &= \$1000\,(1 + 4.21236)\\ &= \$5212.36.\end{aligned}$$

The solution is similar when money is worth 6% nominal, payable semiannually.

34. Perpetuities and capitalization. A *perpetuity* is an annuity whose payments are to be continued forever. Obviously the amount of a perpetuity would be an expression without meaning, since, as time went on, the amount of an annuity would increase beyond all bounds. On the contrary, the present value of a perpetuity is a definite sum, viz. the limit of $a_{\overline{n}|}$ as n increases indefinitely. The notion of a perpetuity is one of great importance in practical business affairs, as we shall soon see.

PROBLEM. *To find the present value of a perpetuity whose payments are made annually.*

Denoting the present value of the perpetuity of 1 per annum by a_∞, we will have, by definition,

$$a_\infty = \lim_{n \doteq \infty} a_{\overline{n}|} = \lim_{n \doteq \infty} \frac{1-v^n}{i}.$$

But

$$v = \frac{1}{1+i}$$

is a fraction whose value is less than unity; hence

$$\lim_{n \doteq \infty} v^n = \lim_{n \doteq \infty} \frac{1}{(1+i)^n} = 0,$$

and, consequently,

$$\lim_{n \doteq \infty} \frac{1-v^n}{i} = \frac{1}{i}.$$

We have therefore

$$a_\infty = \frac{1}{i}. \qquad (1)$$

If the annual rent of the perpetuity is R, the present value will be

$$Ra_{\infty} = \frac{R}{i}. \tag{1'}$$

Equation (1′) shows that an annual income of R per year indefinitely continued is equivalent to a cash sum of $\frac{R}{i}$. For example, if money is worth 5%, an income of \$2000 per year would be equivalent to \$40,000 in cash. Again, an acre of land that can be made to yield a net income of \$5 per annum would be worth \$100 with money at 5%. The cash equivalent of an annual expenditure may be found in the same way. For example, if it costs \$5 a year to dig the weeds from a block of cobblestone gutter, it would pay to spend \$100, in addition to the cost of the cobblestone gutter, to substitute a cement gutter needing no cleaning, when money is worth 5%.

The process of determining the cash equivalent of an annual income or an annual outgo that is supposed to continue indefinitely is called *capitalization*. It is customary to use the expression "to capitalize an income or an outgo" at such and such a rate. In the example relating to the net income of the acre of land, we say that "the annual income of \$5 capitalized at 5% is \$100."

PROBLEM. *To find the present value of a perpetuity whose payments are made at intervals of k years.*

If the first payment be made at the end of k years, the second at the end of $2k$ years, and so on, the sum of the present values of the payments will be

$$v^k + v^{2k} + v^{3k} + \cdots,$$

and if the payments be continued for nk years, the sum may be denoted by $a_{nk,k}$. We have

$$a_{nk,k} = v^k \frac{(1 - v^{nk})}{1 - v^k}$$

as the present value of the first n payments. Dividing the numerator and denominator of the expression on the right by v^k, and remembering that

$$v^k = \frac{1}{(1+i)^k},$$

we have, finally,
$$a_{nk,k} = \frac{1 - v^{nk}}{(1+i)^k - 1}. \tag{2}$$

If now the number of payments be increased indefinitely, nk will be increased indefinitely, and so

$$\lim_{n \doteq \infty} v^{nk} = 0.$$

Denoting $\lim\limits_{n \doteq \infty} a_{nk,k}$ by $a_{\infty,k}$, we have at once

$$a_{\infty,k} = \frac{1}{(1+i)^k - 1}. \tag{3}$$

If, moreover, the payments be R, the present value of the perpetuity will be

$$Ra_{\infty,k} = \frac{R}{(1+i)^k - 1}. \tag{3'}$$

The expressions on the right of (3) and (3′) may be put in forms better adapted for computation by noting that if numerator and denominator be multiplied by i, one factor in the result will be $\frac{1}{s_{\overline{k|}}}$. The equation (3′) then takes the form

$$Ra_{\infty,k} = \frac{R}{i} \cdot \frac{1}{s_{\overline{k|}}}. \tag{3''}$$

In (3″) the value of $\frac{1}{s_{\overline{k|}}}$ can be found from the tables, thus avoiding the laborious division by the value for $(1+i)^k - 1$, which would be necessary if (3) or (3′) were used.

It should be noted that formulas (1) and (3) could have been obtained directly by finding the limit of the sums of the two infinite geometrical progressions

$$v + v^2 + v^3 + \cdots \quad \text{and} \quad v^k + v^{2k} + v^{3k} + \cdots.$$

EXAMPLES

1. If it costs $1000 a year to guard a railroad crossing, how much could the railroad company afford to expend in order to abolish the grade crossing, on the supposition that money could be invested at 4%?

2. A bridge costing $25,000 must be renewed every thirty years. What sum should be set aside when the bridge is first built, to provide for an indefinite number of renewals, on the supposition that the cost of renewal will remain constant and that money will remain at 4%?

Solution. By formula (3″) the amount required will be

$$\frac{25{,}000}{.04}\cdot\frac{1}{s_{\overline{30}|} \text{ at } 4\%}.$$

The tables give $$\frac{1}{s_{\overline{30}|}} = .0178301;$$

consequently, $$\frac{25{,}000}{.04}\cdot\frac{1}{s_{\overline{30}|}} = \$11{,}143.75.$$

The importance of problems like Example 2 becomes evident when we note that the $11,143.75 is *the amount that might be expended over and above the* $25,000, *to make the structure permanent.*

The *capitalized cost* of an article is the first cost plus the present value of the cost of indefinite renewals. If C denote the first cost, C_∞ may be used to denote the capitalized cost. On the supposition that the cost of each successive renewal is the same as the first cost,

$$C_\infty = C + \frac{C}{i}\frac{1}{s_{\overline{k}|}} = C\left(1 + \frac{1}{i}\cdot\frac{1}{s_{\overline{k}|}}\right). \tag{4}$$

When $s_{\overline{k}|}$ is replaced by its value in the second factor on the right, and the resulting expression reduced to a common denominator, it can be shown without difficulty that

$$C_\infty = C\,\frac{(1+i)^k}{i}\cdot\frac{1}{s_{\overline{k}|}}. \tag{5}$$

If C be the cost of a depreciable article used in business, the return to the investor must be estimated on the amount C_∞ and not on the amount C.

Many important problems arising in engineering practice may be solved by making use of the notion of capitalized cost. For

example, a railroad company wishes to find out how much may be expended in treating ties costing 75 cents each laid on the road bed, to extend the life of a tie from seven to fifteen years. It is worth while to solve this important problem in general terms.

PROBLEM. *To determine what amount may be expended upon an article costing C dollars, in order to extend its life from k years to k′ years.*

Let X be the amount to be added to the original cost to put in the more expensive article. The cost of the more expensive article will then be $C+X$. By (5), the capitalized cost of the cheaper article will be

$$C\frac{(1+i)^k}{i}\cdot\frac{1}{s_{\overline{k}|}},$$

while the capitalized cost of the more expensive article will be

$$(C+X)\frac{(1+i)^{k'}}{i}\cdot\frac{1}{s_{\overline{k'}|}}.$$

If in the long run the use of one of the two articles is just as economical as the use of the other, the two capitalized costs will be equal. Consequently,

$$(C+X)\frac{(1+i)^{k'}}{i}\cdot\frac{1}{s_{\overline{k'}|}}=C\frac{(1+i)^k}{i}\cdot\frac{1}{s_{\overline{k}|}}. \tag{6}$$

The required number X is then found by solving (6). Dividing both sides of (6) by

$$\frac{(1+i)^{k'}}{i}\cdot\frac{1}{s_{\overline{k'}|}},$$

and noting that $(1+i)^{-(k'-k)}=v^{k'-k}$, we have

$$C+X=Cv^{k'-k}\cdot s_{\overline{k'}|}\cdot\frac{1}{s_{\overline{k}|}}. \tag{7}$$

Formula (7) gives the first cost of the more expensive article. Transposing C to the right member and taking out the factor $v^{k'-k}$,

$$X=Cv^{k'-k}\left[\frac{s_{\overline{k'}|}}{s_{\overline{k}|}}-(1+i)^{k'-k}\right].$$

To further simplify the result, replace $s_{\overline{k}|}$ and $s_{\overline{k'}|}$ by their values, reduce the second factor to a common denominator, and obtain.

$$X = Cv^{k'-k}\frac{(1+i)^{k'-k}-1}{(1+i)^{k}-1}. \tag{8}$$

If numerator and denominator of the last fraction be divided by i, the result can be placed in the final form

$$X = Cv^{k'-k}\cdot s_{\overline{k'-k}|}\cdot\frac{1}{s_{\overline{k}|}}. \tag{9}$$

We are now ready to solve the crosstie problem proposed above, viz. to find how much a railroad company could afford to pay for treating ties costing 75 cents laid in place, to extend the life from seven to fifteen years. Assuming money to be worth 5%, we have

$$\begin{aligned} X &= \$0.75 \times v^{8} \times s_{\overline{8}|} \times \frac{1}{s_{\overline{7}|}} \\ &= \$0.75 \times .6768 \times 9.5491 \times .1228 \\ &= \$0.595. \end{aligned}$$

Formula (9) can easily be modified to take into consideration the cost of maintenance and other items connected with the use of the article. (See Hickerson, "Formulas for Investment Calculations," reprinted from the *Elisha Mitchell Journal*, Chapel Hill, North Carolina).

EXAMPLES

1. A farm can be made to yield a net annual income of $1000, after improvements are kept up, fertility maintained, and current expenses met. If money is worth 5%, what is the farm worth?

2. A man is offered $10,000 a year for a 99-year lease, or $200,000 cash for a business site when money is worth 5%. Assuming a constant interest rate, which is the better offer? (Solve by inspection.)

3. How much can a railroad company afford to pay to abolish a grade crossing which is guarded by two watchmen, each receiving $600 per year, when money can be invested at 4%?

4. A philanthropist wishes to give an endowment to be used for erecting and keeping in place a building costing $100,000 and requiring to be rebuilt every 50 years. How much money will be required if money is worth 4%?

5. On a certain railroad there is a grade which necessitates the use of a "helper" engine and two crews. If the cost of wages, fuel, oil, and repairs is $10,000 per annum, and a new engine costing $12,000 must be purchased every 20 years, how much could the company afford to expend in reducing the grade to the point where the helper could be dispensed with, money worth 4%?

6. A building costs $100,000 to erect. Annual repairs cost $500. Every 10 years it must be thoroughly overhauled at a cost of $5000, and every 75 years it must be rebuilt at the original cost. What amount of endowment would be necessary in order to build and maintain the structure indefinitely, if money is worth 4%?

7. What amount can be expended in treating a telegraph pole costing $3, to extend its life from 8 to 15 years, if the cost of setting the pole is $5, on the supposition that money is worth 5%?

8. Prove that the amount to be expended in doubling the life of an article is

$$X = Cv^k, \tag{10}$$

where k years is the life of the article without the added expenditure. (Obtain proof by means of formula (8).)

9. If an article of cost C has at the end of its period of service a scrap value S, prove that the amount that may be expended in extending its life from k years to k' years is given by the formula

$$X = v^{k'-k} \cdot s_{\overline{k'-k}|} \cdot \frac{1}{s_{\overline{k}|}} (C - S). \tag{11}$$

10. A man buys a team of horses for $500 and a wagon and harness for $150. The team will have to be replaced in 10 years and the wagon and harness in 20 years. Feed and blacksmith bills cost $200 a year, taxes and insurance $20, and wages of the driver $600. How much must he make in a year in order to realize 8% on his investment, if the investment is capitalized at 5%?

35. Continuous annuities. We may think of an annuity for which the total amount paid in a year is a fixed sum, but the payments are infinitesimally small and are made momently. Such a hypothetical annuity is called a *continuous annuity*. Such annuities do not exist in the actual business world, but they are approximated by the business of large concerns which are receiving many small sums every day.

PROBLEM. *To find the amount of a continuous annuity of* 1 *per annum for n years.*

We will denote the amount of a continuous annuity by $\bar{s}_{\overline{n}|}$, and the present value by $\bar{a}_{\overline{n}|}$. We have

$$\bar{s}_{\overline{n}|} = \lim_{p=\infty} s_{\overline{n}|}^{(p)} = \lim_{p=\infty} \frac{(1+i)^n - 1}{p\left[(1+i)^{\frac{1}{p}} - 1\right]}.$$

As the variable p does not occur in the numerator, we have only to consider the limit of the denominator. But we have already found that

$$\lim_{p=\infty} p\left[(1+i)^{\frac{1}{p}}-1\right] = \log_e(1+i)$$
$$= \delta \qquad \text{(See (3) and (4), § 24.)}$$

where δ is the force of interest. Using these values for the limit in the denominator, we have

$$\bar{s}_{\overline{n}|} = \lim_{p=\infty} s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{\log_e(1+i)} = \frac{(1+i)^n-1}{\delta}. \qquad (1)$$

From the expression for $a_{\overline{n}|}^{(p)}$, we obtain, in exactly the same manner,

$$\bar{a}_{\overline{n}|} = \lim_{p=\infty} a_{\overline{n}|}^{(p)} = \frac{1-v^n}{\log_e(1+i)} = \frac{1-v^n}{\delta}. \qquad (2)$$

Illustrative Example. Find the amount of a continuous annuity of \$1000 per annum for 5 years at 4%.

Solution.
$$\bar{s}_{\overline{5}|} = \frac{(1.04)^5-1}{\log_e(1.04)} = \frac{(1.04)^5-1}{.04 - \frac{(.04)^2}{2} + \frac{(.04)^3}{3} \cdots}$$
$$= \frac{.2166529}{.0392207}$$
$$= 5.52394.$$

The amount of the continuous annuity of \$1000 per annum is \$5523.94.

36. The annuity that will amount to 1.

Problem. *To find the annuity that will amount to* 1 *in n years.*

The problem requires the determination of the annual rent of the annuity that will amount to 1 in n years.

Let R be the annual rent. The amount of an annuity with annual rent R is, by § 31, $Rs_{\overline{n}|}$, or $Rs_{\overline{n}|}^{(p)}$, as the case may be. For an annuity payable annually and amounting to 1 after n years,

$$Rs_{\overline{n}|} = 1;$$

whence

$$R = \frac{1}{s_{\overline{n}|}} = \frac{i}{(1+i)^n - 1}. \qquad (1)$$

If the annuity is payable p times a year, clearly

$$R = \frac{1}{s_{\overline{n}|}^{(p)}} = \frac{p\left[(1+i)^{\frac{1}{p}} - 1\right]}{(1+i)^n - 1}. \qquad (2)$$

The formula for the annual rent of an annuity that will amount to K is found by multiplying equation (1), or equation (2), as the case may be, by K. If the annual rent of an annuity that will amount to K be denoted by R', where

$$R' = KR, \tag{3}$$

then

$$R' = K \cdot \frac{1}{s_{\overline{n|}}} = \frac{Ki}{(1+i)^n - 1}. \tag{4}$$

A similar formula is derived from (2) when the annuity is payable oftener than once a year.

When interest is convertible m times a year, the appropriate formula is found by replacing i and $1+i$, in formula (1), (2), or (4), by their values as given by the fundamental relation

$$i = \left(1 + \frac{j}{m}\right)^m - 1.$$

For reasons that will appear later, the problem of the present section may be called the *sinking-fund problem*, and equation (1), or its equivalent, (4), is called the *sinking-fund equation.* The importance of the problem consists in the fact that it is equivalent to the problem of determining the amount that must be set aside annually to meet a given obligation at the end of a given time. For example, if a city bonds itself for \$200,000 for twenty years, in order to erect a new high school, some provision must be made for the payment of the bonds when they fall due. The most convenient method is to set aside the same amount each year until the debt becomes due. The amount to be set aside each year is found by the formula (1). Substituting the values $K = 200{,}000$, $i = .04$, $n = 20$, we have, as the required amount,

$$\begin{aligned} \$200{,}000 \times \frac{.04}{(1.04)^{20} - 1} &= \$200{,}000 \times .03358175 \\ &= \$6716.35. \end{aligned}$$

The subject will be taken up later, in the sections on sinking funds.

The symbol $\dfrac{1}{s_{\overline{n|}}}$ should be looked upon as the symbol for "the annuity that will amount to 1" rather than as the reciprocal of $s_{\overline{n|}}$.

EXAMPLES

1. A man gives a mortgage for $10,000, to be repaid in 5 years with interest payable annually at 6%. If the interest is paid promptly, what sum must be set aside each year to repay the principal when it falls due, provided the money set aside can be invested at 4%?

2. What sum must be set aside annually to provide for the rebuilding, after 25 years, of a bridge costing $25,000, provided the money set aside can be invested at 4%?

37. The annuity that 1 will purchase. The annuity whose present value is 1 is usually spoken of as "the annuity that 1 will purchase." It plays a large part in the solution of many important problems.

PROBLEM. *To determine the annual rent of an annuity that* 1 *will purchase.*

Let R be the required annual rent. Since the present value of an annuity whose annual rent is R is $Ra_{\overline{n}|}$, the value of will be determined from the equation

$$Ra_{\overline{n}|}=1.$$

Consequently,
$$R=\frac{1}{a_{\overline{n}|}}=\frac{i}{1-v^n} \tag{1}$$

is the required formula. To find the annuity that a given sum A will purchase, it is only necessary to multiply both sides of equation (1) by A. The resulting formula is

$$R'=A\cdot\frac{1}{a_{\overline{n}|}}=\frac{Ai}{1-v^n}. \tag{1'}$$

It is not difficult to determine the formula when interest is convertible oftener than once a year, or when the annuity is payable several times a year.

Formula (1), or its equivalent, (1′), is one form of the *amortization equation* of Euler. It is used in determining the annual installment that will provide for the payment of an interest-bearing debt when principal and interest are to be paid in a series of equal annual installments.

Suppose, for example, a man wishes to pay a debt of \$5000, bearing interest at 6 % effective, in five equal annual installments, the first installment to be paid one year hence. The annual payment is required. The payments constitute an annuity whose annual rent R' is required. By formula (1'),

$$\begin{aligned} R &= \$5000 \cdot \frac{.06}{1-(1.06)^{-5}} \\ &= \$5000 \times .237396 \\ &= \$1186.98. \end{aligned}$$

EXAMPLES

1. Find the formula for the annual rent of an annuity that 1 will purchase when the annuity is payable p times a year and interest is convertible m times a year.

2. A man makes a cash payment of \$2000 on a farm purchased for \$10,000, and wishes to pay the balance, with interest at 5%, in five equal annual installments. What will the annual installment be?

3. Find the formula for the annuity, payable p times a year, for n years, that can be purchased for 1, when interest is at nominal rate j, convertible m times a year.

38. Fundamental relation between $\frac{1}{s_{\overline{n}|}}$ and $\frac{1}{a_{\overline{n}|}}$.

THEOREM. *The annual rent of the annuity that* 1 *will purchase, diminished by the annual rent of the annuity that will amount to* 1, *is equal to* $j_{(p)}$, *where* $j_{(p)}$ *is defined by the equation*

$$j_{(p)} = p\left[(1+i)^{\frac{1}{p}} - 1\right]. \qquad \text{(See (3'), § 23.)}$$

To prove the theorem it must be proved that

$$\frac{1}{a^{(p)}_{\overline{n}|}} - \frac{1}{s^{(p)}_{\overline{n}|}} = j_{(p)}. \tag{1}$$

By §§ 36 and 37,

$$\frac{1}{a^{(p)}_{\overline{n}|}} = \frac{p\left[(1+i)^{\frac{1}{p}} - 1\right]}{1-(1+i)^{-n}} = \frac{j_{(p)}}{1-(1+i)^{-n}},$$

and

$$\frac{1}{s^{(p)}_{\overline{n}|}} = \frac{p\left[(1+i)^{\frac{1}{p}} - 1\right]}{(1+i)^{n}-1} = \frac{j_{(p)}}{(1+i)^{n}-1}.$$

Therefore $\dfrac{1}{a_{\overline{n|}}^{(p)}}-\dfrac{1}{s_{\overline{n|}}^{(p)}}=j_{(p)}\left[\dfrac{1}{1-(1+i)^{-n}}-\dfrac{1}{(1+i)^n-1}\right].$

But
$$\frac{1}{1-(1+i)^{-n}}-\frac{1}{(1+i)^n-1}=\frac{(1+i)^n}{(1+i)^n-1}-\frac{1}{(1+i)^n-1}$$
$$=1.$$

Consequently,
$$\frac{1}{a_{\overline{n|}}^{(p)}}-\frac{1}{s_{\overline{n|}}^{(p)}}=j_{(p)},\tag{1}$$
as was to be proved.

COROLLARY. *When the annuity is payable annually,*
$$\frac{1}{a_{\overline{n|}}}-\frac{1}{s_{\overline{n|}}}=i.\tag{2}$$

To see the truth of (2) we have only to make $p=1$ in (1), since $a_{\overline{n|}}^{(1)}=a_{\overline{n|}}$, $s_{\overline{n|}}^{(1)}=s_{\overline{n|}}$, and $j_{(1)}=1\left[(1+i)^1-1\right]=i.$

Formula (2), which is a special case of the theorem, is the form in which this fundamental relation is most used. As we shall see later, this formula enables us to dispense with a set of tables, either for $\dfrac{1}{a_{\overline{n|}}}$ or for $\dfrac{1}{s_{\overline{n|}}}$, since, when one is given, the formula enables us to find the other at once.

EXERCISE

Prove the relation (2) directly by means of the expressions for $\dfrac{1}{a_{\overline{n|}}}$ and $\dfrac{1}{s_{\overline{n|}}}$.

39. The term of an annuity certain.

PROBLEM. *To find the term of an annuity when the amount, the annual rent, and the rate of interest are given.*

Suppose first that the annuity is payable annually. By formula (1) of § 31, the amount of an annuity of 1 per annum payable annually is
$$s_{\overline{n|}}=\frac{(1+i)^n-1}{i},\tag{1}$$
and consequently the amount of an annuity of R per annum is
$$Rs_{\overline{n|}}=R\frac{(1+i)^n-1}{i}.\tag{2}$$

Let K be the amount of the annuity whose term is sought. Then

$$R\frac{(1+i)^n-1}{i}=K. \tag{3}$$

The required term will be found by solving the equation (3) for n. If equation (3) be divided by R and multiplied by i, and 1 be transposed to the right member,

$$(1+i)^n=1+\frac{Ki}{R}. \tag{4}$$

Equation (4) is an exponential equation which may be solved by taking logarithms of both sides and solving the resulting equation for n. The solution is

$$n=\frac{\log\left(1+\frac{K}{R}i\right)}{\log(1+i)}. \tag{5}$$

If the annuity is payable p times a year, formula (1) is replaced by (3) of § 31, viz.

$$s_{\overline{n}|}^{(p)}=\frac{(1+i)^n-1}{p[(1+i)^{\frac{1}{p}}-1]},$$

and consequently equation (3) will be replaced by

$$R\frac{(1+i)^n-1}{p[(1+i)^{\frac{1}{p}}-1]}=K. \tag{6}$$

From equation (6),

$$(1+i)^n=1+\frac{K}{R}p[(1+i)^{\frac{1}{p}}-1]. \tag{7}$$

Finally, the solution of the exponential equation (7) is

$$n=\frac{\log\left\{1+\frac{K}{R}p[(1+i)^{\frac{1}{p}}-1]\right\}}{\log(1+i)}. \tag{8}$$

If we write $\qquad p[(1+i)^{\frac{1}{p}}-1]=j_{(p)},$

as in (3′) of § 23, formula (8) becomes

$$n=\frac{\log\left(1+\frac{K}{R}\cdot j_{(p)}\right)}{\log(1+i)}. \tag{9}$$

PROBLEM. *To find the term of an annuity when the present value, the annual rent, and the rate of interest are given.*

Suppose the annuity is payable annually, and let A denote the present value. Then, by formula (6) of § 32,

$$R\frac{1-(1+i)^{-n}}{i}=A, \tag{10}$$

where R denotes the annual rent of the annuity. From equation (10),

$$(1+i)^{-n}=1-\frac{Ai}{R}. \tag{11}$$

The solution of the exponential equation (11) for the time n is

$$n=-\frac{\log\left(1-\frac{A}{R}\cdot i\right)}{\log(1+i)}. \tag{12}$$

If the annuity is payable p times a year, equation (10) is replaced by the equation

$$R\frac{1-(1+i)^{-n}}{p[(1+i)^{\frac{1}{p}}-1]}=A. \tag{13}$$

From (13), $\quad (1+i)^{-n}=1-\frac{A}{R}\cdot p[(1+i)^{\frac{1}{p}}-1], \qquad (14)$

and the solution of the exponential equation (14) for the time n is

$$n=-\frac{\log\left\{1-\frac{A}{R}\cdot p[(1+i)^{\frac{1}{p}}-1]\right\}}{\log(1+i)}. \tag{15}$$

If we write $j_{(p)}$ for the expression $p[(1+i)^{\frac{1}{p}}-1]$, according to (3′) of § 23, formula (15) takes the convenient form

$$n=-\frac{\log\left(1-\frac{A}{R}\cdot j_{(p)}\right)}{\log(1+i)}. \tag{16}$$

If the relation between A, R, and i should be such that

$$1-\frac{A}{R}i \not> 0$$

(that is, if R is less than the annual interest charge), the conditions of the problem are inconsistent, since the expression

$$\log\left(1-\frac{A}{R}\cdot i\right),$$

which occurs in formula (12), would then be imaginary, and the time would not be a real number. Suppose, for example, that the annual rent is \$100, the rate is .04, and the present value is assumed to be \$3000. The quantity $\frac{A}{R}i$ would be

$$\frac{3000\times .04}{100}=1.2,$$

and, consequently, $$1-\frac{A}{R}i=-.2,$$

which, as a negative number, has no real logarithm. The significance of this fact is that the present value of an annuity of \$100 per annum at 4% can never equal \$3000, no matter how long the payments may be continued. Indeed, if they were to go on forever, the value of the perpetuity would be $\frac{\$100}{.04}=\$2500$. In short, the problem has no solution unless the annual payment exceeds a year's interest on the present value.

EXAMPLES

1. Find the time in which an annuity of \$1000 per year would amount to \$100,000 if accumulated at 4%.

Solution. By formula (5),

$$\begin{aligned} n &= \frac{\log\left(1+\frac{100{,}000}{1000}(.04)\right)}{\log(1.04)}=\frac{\log 5}{\log 1.04} \\ &= \frac{.6990}{.0170} \\ &= 41.1 \text{ years.}\end{aligned}$$

The result obtained by means of four-place tables of logarithms is only approximate.

2. How long will it take a man to pay for a lot valued at \$2000 with interest at 6%, by paying \$400 at the end of each year?

40. Computation of annuities; the use of annuity tables. In practical work in annuities the term of the annuity and the rate of interest are usually given. We are then required to find the corresponding value of one or more of the four quantities

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}, \tag{1}$$

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \tag{2}$$

$$\frac{1}{s_{\overline{n}|}} = \frac{i}{(1+i)^n - 1}, \tag{3}$$

$$\frac{1}{a_{\overline{n}|}} = \frac{i}{1 - v^n}. \tag{4}$$

Of these four, the first two and either the third or the fourth (usually the fourth) may be found from annuity tables which have been constructed for ordinary rates of interest and for times up to 100 years (see Tables V, VI, and VII). The fourth quantity is easily found by means of the fundamental relation

$$\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} = i. \tag{5}$$

EXAMPLES

1. Find the amount of an annuity of $100 per annum, payable annually for 15 years, with interest at 5% effective.

Solution. From the table of values for $\frac{(1+i)^n - 1}{i}$ we find, opposite the time 15 years, $21.5786 as the amount of 1 per annum. The amount of $100 per annum would be $2157.86.

2. Find the annuity, payable annually for 15 years at effective rate .05, that can be purchased for $5000.

Solution. The annuity that can be purchased for $1.00 is found from the table of values for $\frac{i}{1-v^n}$. For 5% and 15 years we find the value $0.096342. The annuity that $5000 will purchase under the same conditions is

$$\$0.096342 \times 5000 = \$481.71.$$

3. Find the annuity, payable annually, that in 15 years at 5% effective will amount to $5000.

The annuity that will amount to 1 is given by

$$\frac{1}{s_{\overline{n}|}}=\frac{i}{(1+i)^n-1},$$

but no table is given for this expression. However, by the fundamental relation (5) we have

$$\frac{1}{s_{\overline{n}|}}=\frac{1}{a_{\overline{n}|}}-i.$$

Consequently, the annuity that will amount to 1 can be found by subtracting the rate from the annuity that 1 will purchase. In Example 2 we found

$$\frac{1}{a_{\overline{15}|}}=.096342;$$

so that

$$\frac{1}{s_{\overline{15}|}}=.096342-.05$$
$$=.046342.$$

Finally, the annuity that will amount to \$5000 in 15 years at 5% effective is $\$0.046342\times 5000=\$2\cancel{2}1.71.$ 231.71

Tables are not readily accessible for the computation of

$$s_{\overline{n}|}^{(p)},\quad a_{\overline{n}|}^{(p)},\quad \frac{1}{s_{\overline{n}|}^{(p)}},\quad\text{and}\quad \frac{1}{a_{\overline{n}|}^{(p)}}.$$

If, however, the rate of interest is not unusual, the formulas for these quantities can be adapted for computation by means of the subsidiary quantity

$$j_{(p)}=p[(1+i)^{\frac{1}{p}}-1] \tag{6}$$

(see (3′), § 23), or even better by means of the expression

$$\frac{i}{j_{(p)}}=\frac{i}{p[(1+i)^{\frac{1}{p}}-1]}. \tag{7}$$

To see how this may be done, we note that if we multiply numerator and denominator of the expression for $s_{\overline{n}|}^{(p)}$ by i, we have

$$s_{\overline{n}|}^{(p)}=\frac{i}{p[(1+i)^{\frac{1}{p}}-1]}\cdot\frac{(1+i)^n-1}{i}.$$

The first factor on the right is $\dfrac{i}{j_{(p)}}$, and the second is $s_{\overline{n}|}$. Replacing these factors by their values,

$$s_{\overline{n}|}^{(p)} = \frac{i}{j_{(p)}} \cdot s_{\overline{n}|}. \tag{8}$$

It follows immediately that

$$\frac{1}{s_{\overline{n}|}^{(p)}} = \frac{j_{(p)}}{i} \cdot \frac{1}{s_{\overline{n}|}}. \tag{9}$$

In the same way we obtain, from the formula for $a_{\overline{n}|}^{(p)}$,

$$a_{\overline{n}|}^{(p)} = \frac{i}{j_{(p)}} \cdot a_{\overline{n}|}, \tag{10}$$

and from (10),

$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{j_{(p)}}{i} \cdot \frac{1}{a_{\overline{n}|}}. \tag{11}$$

To facilitate computation, tables are constructed for $j_{(p)}$ and $\dfrac{i}{j_{(p)}}$, for $p = 2$, 4, and 12, these being the most common values for p. (See Tables IX and X.)

EXAMPLES

1. Find the amount of an annuity of \$250 per annum, payable quarterly, accumulated for 20 years at 5 % effective.

Solution. The required amount is given by

$$s_{\overline{20}|}^{(4)} = \frac{.05}{j_{(4)} \text{ at } 5\,\%} \cdot s_{\overline{20}|}.$$

From the table of values for $\dfrac{i}{j_{(4)}}$ we find

$$\frac{.05}{j_{(4)} \text{ at } 5\,\%} = 1.01856.$$

Also, from the table for the amount of the annuity of 1 per annum we find

$$s_{\overline{20}|} \text{ at } 5\,\% = 33.065954.$$

Consequently,

$$\begin{aligned} s_{\overline{20}|}^{(4)} \text{ at } 5\,\% &= 1.01856 \times 33.06595 \\ &= 33.67966. \end{aligned}$$

The amount of an annuity of \$250 per annum under the same conditions would be

$$\$33.67964 \times 250 = \$8419.91.$$

2. Find the annuity that will amount to $5000 in 20 years when payable four times a year and accumulated at 5 %.

Solution. We have to find the value of $\frac{1}{s_{\overline{20}|}^{(4)}}$. By formula (9),

$$\frac{1}{s_{\overline{20}|}^{(4)}} = \frac{j_{(4)}}{.05} \cdot \frac{1}{s_{\overline{20}|}}.$$

To find $\frac{1}{s_{\overline{20}|}}$ we use the fundamental relation and obtain

$$\begin{aligned}\frac{1}{s_{\overline{20}|}} &= \frac{1}{a_{\overline{20}|}} - .05\\ &= .0802426 - .05\\ &= .0302426.\end{aligned}$$

From the table of values of $j_{(p)}$ we find

$$j_{(4)} \text{ at } 5\% = .049089;$$

whence
$$\frac{j_{(4)}}{.05} = .98178.$$

Substituting these values in the expression for $\frac{1}{s_{\overline{20}|}^{(4)}}$, we find

$$\begin{aligned}\frac{1}{s_{\overline{20}|}^{(4)}} &= .98178 \times .0302426\\ &= .0296916.\end{aligned}$$

Finally, the annuity that will amount to $5000 is

$$0.0296916 \times 5000 = \$148.46.$$

Another case in which the computations are easily made is that in which the conversion interval for the interest coincides with the interval of payment for the annuity. In this case we have

$$s_{\overline{n}|}^{(p)} = \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{p\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right]}, \qquad (4)\ (\S\ 31)$$

and, making $m = p$, this formula reduces to

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} \cdot \frac{\left(1 + \frac{j}{p}\right)^{np} - 1}{\frac{j}{p}}.$$

The second factor on the right is exactly the expression for the amount of an annuity of 1 per annum, payable annually for np years, i.e. $s_{\overline{np}|}$ at rate $\frac{j}{p}$. We have, then, when $m=p$,

$$s_{\overline{n}|}^{(p)}=\frac{1}{p}\left(s_{\overline{np}|}\text{ at rate }\frac{j}{p}\right). \tag{12}$$

From (12) we have immediately

$$\frac{1}{s_{\overline{n}|}^{(p)}}=p\cdot\frac{1}{s_{\overline{np}|}\text{ at rate }\frac{j}{p}}. \tag{13}$$

Similarly, for $m=p$,

$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\cdot\frac{1-\left(1+\frac{j}{p}\right)^{-np}}{\frac{j}{p}}=\frac{1}{p}\cdot\left(a_{\overline{np}|}\text{ at rate }\frac{j}{p}\right). \tag{14}$$

From (14) it follows directly that when $m=p$,

$$\frac{1}{a_{\overline{n}|}^{(p)}}=p\cdot\frac{1}{a_{\overline{np}|}\text{ at rate }\frac{j}{p}}. \tag{15}$$

EXAMPLES

1. Find the amount of an annuity of $500 per annum, payable June 30 and December 31 of each year, from January, 1900, to January, 1912, if interest is convertible June 30 and December 31 at 6% nominal.

Solution. The time is 12 years, and $m=p=2$. By formula (12),

$$s_{\overline{12}|}^{(2)}=\frac{1}{2}(s_{\overline{24}|}\text{ at rate }.03).$$

From the tables for $s_{\overline{n}|}$,

$$s_{\overline{24}|}\text{ at }3\%=34.4265.$$

Consequently, $$s_{\overline{12}|}^{(2)}=17.2133.$$

Finally, the annuity of $500 per annum, under the same conditions, would be

$$\$17.2133\times 500=\$8606.65.$$

2. Find the annuity which, payable semiannually for 20 years and accumulated at 6% nominal, convertible semiannually, would amount to $10,000

Solution. By formula (13),

$$\frac{1}{s^{(2)}_{\overline{20|}}} = 2 \cdot \frac{1}{s_{\overline{40|}} \text{ at } 3\%}.$$

But
$$\frac{1}{s_{\overline{40|}} \text{ at } 3\%} = \frac{1}{a_{\overline{40|}} \text{ at } 3\%} - .03$$
$$= .043262 - .03$$
$$= .013262.$$

Therefore
$$\frac{1}{s^{(2)}_{\overline{20|}}} = 2 \times .013262$$
$$= .026524,$$

and the annuity that would amount to $10,000 under the same conditions is

$$\$0.026524 \times 10{,}000 = \$265.24.$$

If the rate of interest is unusual, there is nothing to be done but compute the annuity according to the appropriate formula. The computation may be done by means of logarithms, provided a sufficiently extended table is at hand; or, if one does not have the proper table of logarithms, the computation may be abridged by means of the binomial theorem.

Illustrative Example. Find the amount of an annuity of $250 per annum, payable quarterly for 10 years at 5.2% nominal, compounded semiannually.

Solution. The formula for the amount of 1 per annum gives

$$s^{(4)}_{\overline{10|}} = \frac{\left(1 + \frac{.052}{2}\right)^{10} - 1}{4\left[\left(1 + \frac{.052}{2}\right)^{\frac{1}{2}} - 1\right]}.$$

Since the final result is to be multiplied by 250, it will be necessary to compute $s^{(4)}_{\overline{10|}}$ to the fifth decimal place, in order to give a result accurate to the cent.

By the binomial expansion,

$$\left(1 + \frac{.052}{2}\right)^{10} = (1 + .026)^{10}$$
$$= 1 + 10 \times .026 + \frac{10 \cdot 9}{1 \cdot 2}(.026)^2 + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3}(.026)^3 + \cdots$$
$$= 1 + .26 + .03042 + .00210912 + .00009596 \cdots$$
$$= 1.292625.$$

By the ordinary process for extracting the square root,

$$(1.026)^{\frac{1}{2}} = 1.0129165.$$

Consequently, $$4\left[(1.026)^{\frac{1}{2}} - 1\right] = .051666.$$

From these results we obtain

$$s^{(4)}_{\overline{10}|} = \frac{1.292625}{.051666} = 25.0189.$$

The last figure in this result is in question. Finally,

$$\$250 \times s^{(4)}_{\overline{10}|} = \$6254.73.$$

In this result the last figure is valueless and the next one is questionable.

Care should be taken to eliminate unnecessary work in the use of the tables, for, at the best, many of the computations are quite laborious. One means by which unnecessary work may be saved is the dropping of figures on the right of the decimal point in multiplication.

To illustrate, suppose we wish to find the amount of an annuity of \$1257.64 per annum, payable annually for 10 years at 5%.

From the tables,

$$s_{\overline{10}|} \text{ at } 5\% = 12.577893.$$

We have, then, to find the product

$$\$1257.64 \times 12.577893.$$

All the decimals on the right of the multiplier must be used to make sure that the result will be accurate to the cent, since the product

$$1000 \times 12.577893 = 12577.893.$$

The work may then be arranged as follows:

Amount of	\$1000	per annum =	\$12577.893
Amount of	200	per annum =	2515.579
Amount of	50	per annum =	628.895
Amount of	7	per annum =	88.044
Amount of	.60	per annum =	7.542
Amount of	.04	per annum =	.500
Amount of	\$1257.64	per annum =	\$15818.453

The limits of accuracy depend upon the number of decimal places to which the values of the functions $s_{\overline{n}|}$, $a_{\overline{n}|}$, and $\frac{1}{a_{\overline{n}|}}$ are carried. It would, of course, be absurd to carry out results beyond the cents. The number of places required is in general two more than the number of figures required to express the dollars in the sum with which we start. For example, if we wish to find the amount of an annuity of \$275.42 per annum, five-place tables would be required, while if we wished to find the amount of an annuity of \$2750 per annum, it would be necessary to use six-place tables to obtain accurate results.

EXAMPLES

1. Find the amount of an annuity of \$342.14 per annum, payable annually for 21 years and accumulated at 5% effective.

2. If a man saves \$500 a year for 20 years, investing the amount at the end of each year at 6% effective, how much will his savings amount to at the end of the time?

3. A man pays \$60 every half year to a building and loan association that pays $5\frac{1}{2}$% nominal, convertible semiannually. What will his stock be worth in 12 years?

4. A certain preparatory school charges \$800 a year *in advance* for each pupil. What is the cash equivalent for the 4 years' tuition if money is worth 6% effective?

5. A man agrees to pay \$1200 a year at the beginning of each year for 5 years, for a house and lot. If money is worth 6%, what is the cash equivalent for the price paid?

6. A real-estate company has a number of houses upon which it expects to realize \$6000 apiece, and proposes to sell so that the purchaser shall pay a fixed sum at the beginning of each year for 6 years. What annual payment is required if interest be computed at 6% effective?

7. A mine yields \$100,000 a year net, and it is known that it will be worked out in 25 years at the present rate. What is it worth on a 5% basis if it is assumed that the \$100,000 is available at the end of each year?

8. A man buys a farm for \$20,000, paying one half cash. He plans to pay the interest as it falls due, and to set aside at the end of each year a fixed sum which, accumulated at 5%, will enable him to discharge the debt at the end of 10 years. How much will he need to set aside annually?

41. The rate of interest borne by an annuity. Theoretically, any two of the five quantities — rate of interest, term, annual rent, amount, or present value — may be determined, when the other three are given, by means of the two equations, (6) of § 31 and (6) of § 32, viz.

$$K = R\frac{(1+i)^n - 1}{i} \tag{1}$$

$$A = R\frac{1-(1+i)^{-n}}{i}. \tag{2}$$

However, if the rate of interest is the unknown number, serious difficulties are encountered, owing to the fact that the degree of the resulting equation is usually high. The problem of the determination of the rate of interest, though it is among the more important problems with which we have to deal, cannot be solved exactly. The nature of the difficulty and the methods employed in securing an approximation will be illustrated by an example.

Illustrative Example. Let it be required to find the rate of interest at which an annuity of 1 per annum will amount to \$12.50 in ten years.

Solution. Formula (1) becomes

$$12.50 = \frac{(1+i)^{10} - 1}{i}, \tag{3}$$

which, cleared of fractions and transposed, gives

$$(1+i)^{10} - 12.5\,i - 1 = 0. \tag{4}$$

This equation, which is of the tenth degree, cannot be solved by ordinary means. However, from the table for $s_{\overline{n|}}$ we find that i must lie between .045 and .05, since

$$s_{\overline{10|}} \text{ at } 4\tfrac{1}{2}\% = 12.288209, \text{ and } s_{\overline{10|}} \text{ at } 5\% = 12.577893.$$

If we put

$$i = .05 + h, \tag{5}$$

we know that h is numerically very small and negative. If the value of i from (5) be substituted in (4), the resulting equation is

$$(1.05 + h)^{10} - 12.5\,h - 1.625 = 0. \tag{6}$$

Expanding $(1.05 + h)^{10}$ by the binomial theorem, and substituting the expansion in (6), we obtain

$$(1.05)^{10} + 10 \times (1.05)^9 h + 45 \times (1.05)^8 h^2 + \cdots + h^{10} - 12.5\,h - 1.625 = 0. \tag{7}$$

Now, since h is numerically very small, certainly not greater in absolute value than .005, the terms in $h^2, h^3, \cdots, h^{10}$ are relatively insignificant, and we shall obtain an approximate value for it by dropping all powers above the first. The linear equation for the approximate determination of h is therefore

$$(1.05)^{10} + 10 \times (1.05)^9 h - 12.5 h - 1.625 = 0. \tag{8}$$

From equation (8),

$$\begin{aligned} h &= \frac{1.625 - (1.05)^{10}}{10\,(1.05)^9 - 12.5} \qquad (9)\\ &= \frac{-.003895}{3.013282} \\ &= -.0013. \end{aligned}$$

Consequently, $i = .05 - .0013 = .0487$ approximately.

Assuming that .0487 is a better approximation than .05, as is really the case, we may write

$$i = .0487 + h',$$

where h' is smaller than h. This value of i may be substituted in equation (4), and an approximate value of h' may be found which will give a still closer approximation for i. Using the value .0487 for i instead of .05, we find, from (4),

$$(1.0487 + h')^{10} - 12.5\,(.0487 + h') - 1 = 0.$$

Expanding the power of the binomial, dropping powers of h' above the first, and solving the resulting equation for h',

$$\begin{aligned} h' &= \frac{1.60875 - (1.0487)^{10}}{10 \cdot (1.0487)^9 - 12.5} \qquad (10)\\ &= -.000032. \end{aligned}$$

The approximate value of i corresponding to this value of h' will therefore be

$$i = .048668.$$

The error in this result is less than .000002, so that the approximation is sufficiently exact for all practical purposes.

The method is general for this particular class of problems and applies equally well to problems in which it is required to find the rate of interest from the equation for the present value, viz.

$$a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i}.$$

The method may be formulated into the following rule:

To find the rate of interest for an annuity when the annual rent, the time, and the amount (or the present value) are given, find from the tables the two values between which i lies; select an approximate

value, i', lying between these two, and in the appropriate formula substitute $i' + h$ for i. Integralize the resulting equation, expand all powers of $1 + i' + h$ into a power series in h, drop powers of h above the first, and solve the resulting linear equation in h. The value

$$i' + h$$

*will then be a first approximation. If a closer approximation is desired, repeat the process, using $i' + h$ instead of i'.**

MISCELLANEOUS EXAMPLES AND PROBLEMS

1. A mine yields $5000 net payable to the owner in two equal semiannual installments at the middle and the end of the year. If the payments are accumulated at 5% nominal, convertible semiannually, what will the amount be at the end of 20 years?

2. Find the amount in the previous problem if the payments are accumulated at 6% effective.

*Students who are familiar with the theory of equations may object to the method of approximation given above, for the reason that the equation whose roots we wish to find is usually of high degree, and among its many roots we cannot be sure that the successive approximations will converge toward the one desired. A little consideration will show that there is no danger at this point.

Suppose we are finding i from the equation for the amount of the annuity, viz.

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}, \tag{1}$$

or, what amounts to the same thing, from the equation

$$(1+i)^n - s_{\overline{n|}}\, i - 1 = 0. \tag{2}$$

If z be substituted for $1 + i$, i.e. if $i = z - 1$, equation (2) takes either the form

$$z^n - s_{\overline{n|}}\, z + (s_{\overline{n|}} - 1) = 0 \tag{3}$$

or the form

$$z^n - 1 - s_{\overline{n|}}(z - 1) = 0. \tag{4}$$

The form (3) is an equation with three terms, and as such it cannot have more than two changes of sign in its coefficients. Consequently, by Descartes's rule of signs, the equation cannot have more than *two real positive roots*. Moreover, the form (4) shows that $z = 1$ is one root, so that there cannot be more than one other which is real and positive. The root $z = 1$ gives $i = 0$. The other root whose approximate value we are seeking is therefore separated by a considerable amount from the root $i = 0$. Consequently, there is little danger of confusing two roots which are nearly equal in value.

Similar considerations apply to the equation

$$a_{\overline{n|}} = \frac{1 - (1+i)^{-n}}{i},$$

which may be reduced to either the form

$$a_{\overline{n|}}\, z^{n+1} - (a_{\overline{n|}} + 1)\, z^n + 1 = 0$$

or the form

$$a_{\overline{n|}}\, z^n (z - 1) - (z^n - 1) = 0.$$

3. A mine yielding $2500 net at the end of each half year will be "worked out" in 20 years, and the equipment will be worth $1000 at that time. What is the value of the mine if money is worth 6%?

4. A man gives a mortgage to secure a loan on his property. How much must he invest at the end of each year, in a savings bank that pays 4% effective, in order to discharge the mortgage at the end of 10 years, provided interest is paid as it falls due?

5. If in the previous problem the payments are made to the savings bank at the end of each half year, and interest is convertible half yearly at nominal rate .04, what would the half-yearly payment be?

6. A man pays $20 per month for 12 years into a building and loan association which pays $5\frac{1}{2}$% nominal, convertible semiannually. What will his stock be worth at the end of the time?

7. A life-insurance policy matures with a value of $5000, and the insurance company gives the option between the whole amount in cash and ten equal annual installments, the first to be paid one year after the maturity of the policy. What will be the annual installment if the interest rate is .04?

8. A father, wishing to provide for the education of his son, sets aside, on the day of the son's birth, a sum which, invested at 5%, will yield $500 a year for 4 years, the first installment to be paid on the boy's seventeenth birthday. What is the sum to be set aside?

9. What sum must be paid each month to a building and loan association which pays 5.75% nominal, convertible semiannually, to enable a man to discharge a debt of $4000 in 12 years?

10. On a 5% basis, what is the value of an acre of land that produces a net return of $8 per year?

11. A railroad company employs 6 men at $600 each, to guard its crossings in a certain city. How much could it afford to spend for track elevation if money is worth 5%?

12. A man buys a piano for $500, agreeing to pay $20 a month with interest at 6% effective on all sums remaining due. If the first payment is made on the date of purchase, how long will it take to pay for the instrument?

13. A bridge costing $25,000 new must be replaced after 25 years. If money can be invested at 5% effective, what sum should be set aside annually to provide for the rebuilding of the bridge, provided it can be rebuilt for the same amount as the cost of the original structure?

14. How long will it take a man to accumulate $100,000 by saving $1000 a year and investing it at 6% effective?

15. What is the rate of interest on an annuity of $100 per year, payable annually, which amounts to $1300 in 10 years?

16. The present value of an annuity of \$100 per year, payable annually for 10 years, is \$780. What is the rate?

17. The amount of an annuity of \$100 per annum, payable annually, is \$1300 and the present value is \$750. What is the rate of interest?

Suggestion. Use the fundamental relation between $\frac{1}{a_{\overline{n}|}}$ and $\frac{1}{s_{\overline{n}|}}$, or solve directly by eliminating n between the equations giving the values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

18. Prove that
$$s_{\overline{n}|}^{(p)} = (1+i)^n a_{\overline{n}|}^{(p)},$$
and interpret the result in words.

19. Find the relation
$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$
by eliminating n from the two equations
$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \quad \text{and} \quad a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i}.$$

20. Prove that when the conversion interval for interest coincides for the interval of payment for the annuity,
$$a_{\overline{n}|}^{(p)} = \frac{1}{p} a_{\overline{np}|} \text{ at rate } \frac{j}{p}. \qquad \text{(See (12), § 40.)}$$

21. Prove that when the conversion interval and the interval of payment coincide,
$$s_{\overline{n}|}^{(m)} = s_{\overline{n}|} \cdot \frac{i}{j},$$
where $s_{\overline{n}|}$ is computed at the effective rate i.

22. Prove that
$$a_{\overline{n}|}^{(m)} = a_{\overline{n}|} \cdot \frac{i}{j}$$
where $a_{\overline{n}|}$ is computed at the effective rate i.

23. Prove that when the conversion interval coincides with the interval of payment,
$$\frac{1}{s_{\overline{n}|}^{(p)}} = \frac{p}{s_{\overline{np}|}} \text{ at rate } \frac{j}{p},$$
and
$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{p}{a_{\overline{np}|}} \text{ at rate } \frac{j}{p}.$$

24. Find the expression for an annuity of 1 per annum payable every 5 years for n years (a) when n is an exact multiple of 5; (b) when n contains 5 q times with a remainder r.

25. If $\mathbf{s}_{\overline{n}|}$ denote the amount of "annuity due" at the end of n years, prove that
$$\mathbf{s}_{\overline{n}|} = s_{\overline{n-1}|} + 1.$$

CHAPTER VII

THE EXTINCTION OF INTEREST-BEARING DEBTS BY PERIODICAL PAYMENTS — AMORTIZATION

42. The amortization of an interest-bearing debt by equal annual installments. In finance the term *amortization* frequently means the extinction of a debt, regardless of the means employed; in the present section it is used in connection with a particular method, viz. the repayment of a debt, principal and interest, in equal annual installments. A bond is issued or a loan is made, and it is agreed that a fixed amount, which must be in excess of a year's interest, will be paid each year. Of this fixed amount a portion goes to the payment of interest due, and the balance is applied toward the reduction of the principal. For example, a savings bank makes a loan of $1000, on which 6% is to be paid, and it is arranged that $100 shall be paid each year. At the end of the first year an interest payment of $60 will be due, leaving $40 to apply toward the reduction of the principal. The principal for the second year will be $960, and the interest payment will be $57.60, so that $42.40 will be available for the reduction of the principal. The process continues in this way (in this case through some sixteen years) until the debt is entirely repaid. The method has obvious advantages for the debtor.

In the example just given, the amount of the annual payment was given, and the time required to pay the debt was unknown. Frequently the time is given and the amount of the annual payment is the unknown. There are, then, two fundamental problems in connection with this sort of amortization: viz. to find the annual installment when the principal and the time are known, and to find the time when the principal and the annual payment are known.

PROBLEM. *To determine the periodical payment required to extinguish a given interest-bearing debt in a given time.*

Let A denote the face of the debt, n the time, and i the rate of interest. The several payments will constitute an annuity whose present value must equal the debt. Let R denote the annual payment, which is therefore the annual rent of the annuity whose present value is A. By equation (6) of § 32,

$$A = Ra_{\overline{n}|}.$$

Consequently,
$$\boldsymbol{R = A \cdot \frac{1}{a_{\overline{n}|}} = \frac{Ai}{1-(1+i)^{-n}}}. \tag{1}$$

If the payment is made p times a year,

$$A = Ra_{\overline{n}|}^{(p)},$$

and
$$R = A \cdot \frac{1}{a_{\overline{n}|}^{(p)}} = A \cdot \frac{p[(1+i)^{\frac{1}{p}}-1]}{1-(1+i)^{-n}}. \tag{2}$$

If we multiply numerator and denominator of (2) by i, we have

$$R = \frac{A}{i} \cdot j_{(p)} \cdot \frac{1}{a_{\overline{n}|}}. \tag{2'}$$

The periodical payment will be

$$\frac{R}{p} = \frac{A}{pi} \cdot j_{(p)} \cdot \frac{1}{a_{\overline{n}|}}. \tag{3}$$

Both (2′) and (3) are adapted for the use of logarithms.

PROBLEM. *To find the time required to extinguish a given interest-bearing debt by means of equal annual installments of given amount.*

The unknown number n is found by solving equation (1), and the result, as given by (12) of § 39, is

$$n = -\frac{\log\left(1-\frac{A}{R}i\right)}{\log(1+i)}. \tag{4}$$

When payments are made p times a year, the value of n is given by either (15) or (16) of § 39. We have, by (16) of § 39,

$$n = -\frac{\log\left(1-\frac{A}{R}\cdot j_{(p)}\right)}{\log(1+i)}. \tag{5}$$

EXAMPLES

1. What annual installment will extinguish a debt of \$1000, bearing 6% interest, in 15 years?

Solution. In this problem $A = \$1000$, $n = 15$, and $i = .06$.

Consequently, $$\$1000 = R \cdot a_{\overline{15|}},$$

and $$R = \$1000 \frac{1}{a_{\overline{15|}}}.$$

By the tables, $$\frac{1}{a_{\overline{15|}}} \text{ at } 6\% = .10296,$$

so that $$R = \$1000 \frac{1}{a_{\overline{15|}}} = \$102.96.$$

2. A man buys a piano for \$500, agreeing to pay \$100 down and the balance in equal monthly installments of \$20 with interest at 6%. How long will it take him to complete the payment?

Solution. By (16), § 39, we have

$$n = -\frac{\log\left(1 - \frac{400}{240}(.06)_{(12)}\right)}{\log(1.06)}.$$

From the table of values for $j_{(p)}$,

$$.06_{(12)} = .05841,$$

so that

$$\begin{aligned} n &= -\frac{\log .90265}{\log 1.06} \\ &= -\frac{9.955519 - 10}{.025306} \\ &= \frac{.044481}{.025306} \\ &= 1.75 \text{ years, approximately.} \end{aligned}$$

3. What is the monthly payment required to pay for a piano costing \$500, with interest at 6%, in 5 years.

Solution. In this problem we have

$$A = \$500,\ n = 5,\ i = .06, \text{ and } p = 12.$$

By formula (3), $$\text{monthly payment} = \frac{R}{12} = \frac{500}{12 \times .06} \cdot (.06)_{(12)} \frac{1}{a_{\overline{5|}}}.$$

The tables give $$(.06)_{(12)} = .05841$$

and $$\frac{1}{a_{\overline{5|}}} \text{ at } 6\% = .23740.$$

By logarithms we find $$\frac{R}{12} = \$9.63.$$

4. A man buys a farm, making a cash payment and leaving unpaid the sum of $5000, which he agrees to pay, principal and interest at 6 %, in five equal annual installments, the first to be paid one year hence. What will be the amount of the annual installment?

5. Find the annual payment required to extinguish a debt of $5000 with interest at 6 % if the first of the equal annual payments is cash.

6. Prove formula (1) by first accumulating the annuity to the end of the term and then finding the annuity that will amount to the accumulation.

7. What monthly payment, continued for 10 years, will suffice to repay a loan of $4000, principal and interest at 6 %, the first payment to be made one month from the date on which the debt was incurred?

8. A city having an assessed valuation of $30,000,000 votes bonds to the amount of $200,000 for the erection of a high-school building, and arranges to pay principal and interest at 6%, payable annually, in twenty equal annual installments. How much will the rate of taxation be increased, and how much will the tax of a man whose property is assessed at $40,000 be increased?

43. The amount remaining due after the *r*th payment has been made. When the annual payment required to extinguish a debt in a given time has been determined by formula (1) or formula (2) of the previous section, the payments could be made regularly until the debt is paid, thus ending the transaction. It is, however, of great importance that the exact relation of debtor to creditor should be accurately known at any given time while the process of extinguishing the debt is going on. The city clerk, for example, must be able at any time to state exactly the amount of the city's indebtedness.

PROBLEM. *To find the amount remaining due after the rth payment has been made.*

The principles involved are of such importance that it is worth while to show the whole process in detail, and to obtain a solution by two methods.

First solution. Let R be the annual payment as determined by § 42, and let A be the principal to be repaid. Also, let

$$A_1, \quad A_2, \quad A_3, \quad \cdots, \quad A_r, \quad \cdots, \quad A_n,$$

be the amounts remaining due after the first, second, $\cdots$, and rth payments have been made.

At the end of the first year the amount due will be $A(1+i)$, and the amount remaining due after the first payment has been made will be

$$A_1 = A(1+i) - R.$$

For the second year the debt is A_1, and the amount remaining due after the second payment has been made is

$$A_2 = A_1(1+i) - R.$$

Similarly, $$A_3 = A_2(1+i) - R,$$

and, in general, $$A_r = A_{r-1}(1+i) - R.$$

But if we replace A_1, in the second of these equations, by its value as given in the first,

$$\begin{aligned} A_2 &= [A(1+i) - R](1+i) - R \\ &= A(1+i)^2 - [(1+i) + 1]R. \end{aligned}$$

This value for A_2, substituted in the third, gives

$$\begin{aligned} A_3 &= \{A(1+i)^2 - [(1+i)+1]R\}(1+i) - R \\ &= A(1+i)^3 - [(1+i)^2 + (1+i) + 1]R. \end{aligned}$$

Proceeding in this fashion, we shall find

$$A_r = A(1+i)^r - [(1+i)^{r-1} + (1+i)^{r-2} + \cdots + (1+i) + 1]R.$$

But the expression in the bracket is precisely the expression for $s_{\overline{r}|}$ as found in § 31. Therefore

$$\mathbf{A_r = A(1+i)^r - Rs_{\overline{r}|}.} \tag{1}$$

The value of A_r is readily found by means of (1) with the help of a table of compound amounts and a table of amounts of an annuity of 1 per annum.

Second solution. A second and much simpler solution is readily obtained by noting that at any point in the process the remaining payments constitute an annuity. After the rth payment has been made, there will be $n-r$ payments to make. These $n-r$ payments of R per annum constitute an annuity with a term of $n-r$ years and a present value $Ra_{\overline{n-r}|}$. But this present value should be precisely the amount remaining due after the rth payment has been made. It follows that

$$\mathbf{A_r = Ra_{\overline{n-r}|}.} \tag{2}$$

Formula (2) is better than formula (1), since it enables one to find the amount remaining due by means of a single table, viz. the table of present values of an annuity of 1 per annum.

A second solution of the problem of determining the annual payment R is readily obtained by means of equation (1). This equation is true for all values of R from 1 to n inclusive. But clearly $A_n = 0$, so that when $r = n$, equation (1) becomes

$$A(1+i)^n - Rs_{\overline{n}|} = 0,$$

or

$$A(1+i)^n - R\frac{(1+i)^n - 1}{i} = 0. \tag{3}$$

From equation (3), which is Euler's *amortization equation* (see § 37),

$$R = \frac{Ai(1+i)^n}{(1+i)^n - 1}.$$

Dividing numerator and denominator by $(1+i)^n$, we obtain

$$R = \frac{Ai}{1-(1+i)^{-n}} = \frac{Ai}{1-v^n} = A \cdot \frac{1}{a_{\overline{n}|}}, \tag{4}$$

which is identical with (1) of § 37 and (1) of § 42.

EXAMPLES

1. A man arranges to pay a debt of \$10,000, with interest at 5 %, payable annually, in ten equal annual installments, but after paying six installments, obtains permission to pay the balance remaining due on the date when he would have paid the seventh installment. How much does he have to pay at this time?

First solution. By (1) of § 42, the annual installment is

$$\begin{aligned} R &= \$10{,}000\left(\frac{1}{a_{\overline{10}|}} \text{ at } 5\,\%\right) \\ &= \$10{,}000 \times .129505 \\ &= \$1295.05. \end{aligned}$$

Consequently, by (1), the amount remaining due after the seventh installment has been paid would be

$$\begin{aligned} A_7 &= \$10{,}000(1.05)^7 - \$1295.05\,(s_{\overline{7}|} \text{ at } 5\,\%) \\ &= \$10{,}000 \times 1.407100 - \$1295.05 \times 8.142008 \\ &= \$14071.00 - 10544.31 \\ &= \$3526.69. \end{aligned}$$

To this amount must be added the seventh payment, which is to be paid at the same time. The total amount to be paid is therefore \$4821.74.

Second solution. The annual payment was found by the first solution to be \$1295.05. By formula (2) the amount remaining due after the seventh installment is paid will be

$$\begin{aligned} A_7 &= R\,(a_{\overline{8}|} \text{ at } 5\%) \\ &= \$1295.05 \times 2.723248 \\ &= \$3526.74. \end{aligned}$$

The amount due *just before* the seventh installment would have been paid is therefore

$$\$3526.74 + \$1295.05 = \$4821.79.$$

The discrepancy of 5 cents in the two results is due to the fact that sufficiently extended tables have not been used.

2. A debt of \$100,000 is to be paid, principal and interest at 5%, in 20 equal annual installments, the first installment due in 1 year. What is the annual payment, and what remains due after the tenth payment is made?

3. On January 1, 1905, a city borrowed, for the construction of a system of waterworks, \$400,000, to be repaid, principal and interest at 5%, payable annually, in 20 equal annual installments, the first installment to be paid January 1, 1906. What amount will remain due immediately after the payment is made on January 1, 1915?

4. Suppose a railway company were to borrow \$1,000,000, to be repaid, principal and interest at 5%, in 20 equal annual installments, the first installment to be paid *five years* after the date of making the loan. What would the amount of the annual installment be?

5. Prove that the two values for A_r, viz.

$$A_r = A\,(1+i)^n - Rs_{\overline{r}|}$$

and

$$A_r = Ra_{\overline{n-r}|},$$

as given by formulas (1) and (2), are identical.

Suggestion. Replace A by $Ra_{\overline{n}|}$ or R by $A \cdot \dfrac{1}{a_{\overline{n}|}}$.

6. Suppose the face value of the debt A, the annual payment R, and the rate of interest are given. Find the expression for the time from the amortization equation (3).

7. Find the amortization equation when p payments, whose aggregate amount is R, are made in a year.

8. Find the formula for the amount remaining due after the rth payment is made on a debt bearing interest at effective rate i, if a payment R is made regularly every pth part of a year.

44. The amount remaining due when both the payments and the time intervals are unequal.

PROBLEM. *To find the amount remaining due on a debt bearing interest convertible annually, after several unequal payments have been made at unequal intervals of time.*

Let $R_1, R_2, \cdots$ be the several payments made at times $t_1, t_2, \cdots$ after the debt was incurred, and let $A_1, A_2, \cdots$ be the amounts remaining due after the first, second, $\cdots$ payments are made.

Employing the first method used in the solution of the previous problem,
$$A_1 = A(1+i)^{t_1} - R_1.$$

The time interval between the first and second payments will be $t_2 - t_1$. Consequently,
$$\begin{aligned} A_2 &= A_1(1+i)^{t_2-t_1} - R_2 \\ &= A(1+i)^{t_2} - R_1(1+i)^{t_2-t_1} - R_2. \end{aligned}$$

Similarly, $A_3 = A(1+i)^{t_3} - R_1(1+i)^{t_3-t_1} - R_2(1+i)^{t_3-t_2} - R_3.$

In general,
$$A_r = A(1+i)^{t_r} - R_1(1+i)^{t_r-t_1} - R_2(1+i)^{t_r-t_2} - \cdots - R_r. \quad (1)$$

The problem of the present section is exactly the problem of partial payments when interest is convertible annually. The student should translate the formula (1) into words, and thus obtain a rule for the solution of problems.

EXAMPLE

A man bought a farm for $10,000, paying $3000 in cash, giving a mortgage note for the balance with interest at 6%, payable annually. He paid $2800 at the end of the first year, $1500 at the end of the second, $2000 at the end of the third, and the balance at the end of the fourth year. What was the amount of the last payment?

45. The amortization schedule. For the purpose of keeping a record of such transactions as have been described in §§ 42 and 43, it is advisable to construct a table which will show the following things: the principal remaining due, or *outstanding*, immediately after the annual payment, the interest for the interval, and the amount of principal repaid. Such a table is called

an *amortization schedule.* Such a schedule, or its equivalent, is almost a necessity in the bookkeeping of any institution which has to do with the amortization of debts by equal annual installments. The construction will be most easily understood by examining a concrete case.

Let it be required to construct a schedule showing the amortization of a loan of \$10,000, with interest at 5%, in five equal annual installments.

By solving the amortization equation, or by equation (1) of § 42, we find the expression for the annual installment, viz.

$$R = A \cdot \left(\frac{1}{a_{\overline{5|}}} \text{ at } 5\%\right).$$

From the table for the annuity that 1 will purchase,

$$\frac{1}{a_{\overline{5|}}} \text{ at } 5\% = 0.230975.$$

Consequently, the annual payment is

$$R = \$2309.75.$$

Of this amount \$500 must go to pay interest due, leaving \$1809.75 to apply to the reduction of the principal. For the second year the principal outstanding will therefore be \$8190.25, and the interest on this sum for the year at 5% will be \$409.51, so that, after interest is paid, there will be available from the second installment \$1900.24 for the reduction of the principal. The remaining parts of the schedule are computed in a similar manner.

Schedule for the amortization of a debt of \$10,000, *with interest at* 5%, *payable annually, in* 5 *equal annual installments:*

Year	Principal outstanding at beginning of year	Interest at 5%	Annuity	Principal repaid
1	\$10,000.00	\$500.00	\$2,309.75	\$1,809.75
2	8,190.25	409.51	2,309.75	1,900.24
3	6,290.01	314.50	2,309.75	1,995.25
4	4,294.76	214.74	2,309.75	2,095.01
5	2,199.75	109.99	2,309.75	2,199.76
	\$30,974.77	\$1,548.74	\$11,548.75	\$10,000.01

Several checks upon the accuracy of the work are possible. The first and most important is that the sum of the amounts in the column for "principal repaid" should be equal to the original principal. In the above schedule the slight discrepancy is caused by the fact that the tables employed are not sufficiently extended. Seven-place tables would give, for the annuity that 1 would purchase for five years at 5%, \$0.2309748, and if this value were used, the results would check exactly.

In the second place, the total interest paid should equal the interest for one year on the amount found by adding together the amounts in the column of "principal outstanding." Finally, the total interest plus the original principal should be equal to the total sum paid, i.e. five times the annual installment.

EXAMPLE

Construct the schedule for the amortization of a loan of \$2000, at 6%, to be repaid in 5 equal annual installments.

46. The amortization schedule in general terms. If we look upon the debt as the present value of an annuity of R per annum, we may easily construct the schedule in general terms. To do this, let R denote the annual payment. The principal outstanding at the beginning of the first year is then

$$Ra_{\overline{n}|} = R\frac{1-v^n}{i}; \tag{1}$$

the interest due for the first year is

$$Ra_{\overline{n}|}\, i = R(1-v^n); \tag{2}$$

the principal repaid is, the annual payment diminished by the interest, or,

$$R - R(1-v^n) = Rv^n; \tag{3}$$

and, finally, the principal remaining unpaid, or the principal outstanding for the second year, is

$$\begin{aligned} Ra_{\overline{n}|} - Rv^n &= R\left[\frac{1-v^n}{i} - v^n\right] \\ &= R\left[\frac{1-(1+i)v^n}{i}\right]. \end{aligned}$$

But $$(1+i)v^n = v^{n-1},$$

since $$(1+i) = \frac{1}{v}.$$

We have, then, $$Ra_{\overline{n}|} - Rv^n = R\frac{1-v^{n-1}}{i} = Ra_{\overline{n-1}|}. \qquad (4)$$

Starting with $Ra_{\overline{n}|}$ as the amount of the debt, we have for the first line of the table the numbers

$$Ra_{\overline{n}|}, \quad R(1-v^n), \quad R, \quad Rv^n.$$

If we take $Ra_{\overline{n-1}|}$ as the principal outstanding at the beginning of the second year, the other quantities in the line may be computed in the same way as the quantities in the first line, or, better yet, they may be written down by analogy with the first line. The remaining lines present no difficulty. The schedule is, then, as follows:

Schedule for the amortization of a debt of $Ra_{\overline{n}|}$ in n equal annual installments:

Number of interval	Principal outstanding at beginning of interval	Interest for interval	Annuity	Principal repaid at end of interval
1	$Ra_{\overline{n}|}$	$R(1-v^n)$	R	Rv^n
2	$Ra_{\overline{n-1}|}$	$R(1-v^{n-1})$	R	Rv^{n-1}
3	$Ra_{\overline{n-2}|}$	$R(1-v^{n-2})$	R	Rv^{n-2}
...	...	...	...	...
n	$Ra_{\overline{1}|}$	$R(1-v)$	R	Rv

It is important to note, from the schedule in general terms, that, for the particular case, the greater part of the schedule could be derived almost directly from the annuity tables. The amounts in the column for principal outstanding would be obtained from a table for the present value of an annuity, while the v^n occurring in two of the other columns is nothing but the present value of 1 for n years. It is of interest to note also that the amounts of principal repaid in successive years form a geometrical progression with ratio $\frac{1}{v}$.

47. The amortization schedule when the debt is expressed in bonds of a given denomination. When a debt is repaid by the retirement of bonds of a given denomination, the amount of principal repaid at a given time must be a multiple of the denomination of the bonds. For example, if the denomination of the bonds is \$1000, the principal repaid must be \$1000 or \$2000 or some other multiple of \$1000. One could not pay \$2375 on the principal. In such cases the annual payments are made (as nearly as possible) equal to the amount required to pay the debt in equal annual installments. An example will make the matter clear.

Suppose it is required to construct a schedule for the amortization, in 10 years, of a loan of \$100,000 at 5%, consisting of 100 bonds, each of denomination \$1000, with annual payments as nearly equal as possible.

The amount that would be required to repay such a loan in equal annual installments is found by the amortization equation to be \$12,950.46, and of this amount \$5000 must go to the payment of interest, leaving \$7950.46 available for the reduction of the principal on the basis of equal annual payments. But since only whole bonds can be retired, \$8000 is the amount nearest to \$7950.46 that it is possible to pay.

The annual payment would then be \$13,000, and the principal unpaid would be \$92,000. The second year's interest will be \$4600, and the amount available for repayment of principal about \$8300, so that the payment on the principal should be again \$8000. The remainder of the schedule explains itself.*

* When a loan expressed in terms of bonds of a given denomination is to be repaid gradually over a series of years, the determination of the precise bonds that are to be repaid each year is a matter that must be considered. This determination is usually effected by what are called drawings. The bonds are numbered consecutively, and, at some convenient time before a payment is to be made, the numbers of the bonds remaining unpaid are put into a box and thoroughly shaken; numbers are then drawn out one by one, until a sufficient number have been drawn to exhaust the funds available after the interest has been paid. The numbers thus drawn are published, together with the statement that, after a certain date, interest will cease and the principal named in the bond will be repaid. This method is frequently followed in the case of European loans, though it has not as yet found much favor in this country, possibly for the reason that the time to elapse before the bond is to be redeemed, which is an element in its valuation, cannot be determined in advance (see Chapter VIII).

Schedule for the repayment of a loan of $100,000, *expressed in bonds of* $1000, *in* 10 *annual installments as nearly equal as possible:*

Year	Principal outstanding	Interest at 5%	Annual payment	Principal repaid	Number of bonds retired
1	$100,000	$5,000	$13,000	$8,000	8
2	92,000	4,600	12,600	8,000	8
3	84,000	4,200	13,200	9,000	9
4	75,000	3,750	12,750	9,000	9
5	66,000	3,300	13,300	10,000	10
6	56,000	2,800	12,800	10,000	10
7	46,000	2,300	13,300	11,000	11
8	35,000	1,750	12,750	11,000	11
9	24,000	1,200	13,200	12,000	12
10	12,000	600	12,600	12,000	12
	$590,000	$29,500	$129,500	$100,000	100

The student who is interested in amortization plans should consult Schlimbach's "Politische Arithmetik," Frankfurt am Main, 1902. In this book no less than twenty-seven different plans are given.

EXAMPLES

1. A city makes a loan of $400,000, to be repaid, with interest at 5%, in 20 equal annual installments. Construct a schedule.

2. Construct a schedule for the repayment of a loan of $1,000,000, with interest at 4%, to be repaid in 20 annual installments as nearly equal as possible if the bonds are of denomination $1000.

3. A man borrows $2000 at 6% interest, and agrees to repay it at the rate of $~~240~~ 600 a year. Construct the schedule.

Note. When loans are repayable in this manner, it is customary to say that the loan bears interest at 6% *with* 6% *for amortization.*

√ **4.** Construct the schedule for the repayment of a loan of $4000 bearing 6% interest with 6% for amortization.

5. Construct the schedule for the repayment of a loan of $4000 bearing 5% interest with 6% for amortization.

6. Construct the general amortization schedule when the principal is payable in *np* equal installments to be paid *p* times a year.

CHAPTER VIII

THE VALUATION OF BONDS

48. Bonds bought to yield a certain rate of interest. For present purposes a *bond* may be defined as a certificate of ownership in a definite portion of a debt due from a government, a city, a business corporation, or an individual. In its simplest form it is a promise to pay a stipulated sum on or after a given date, and to pay interest or dividends at a specified rate and at definite intervals. The interval between interest payments is either a year, a half year, or a quarter year. The amount named in the bond is called the *par value*, or sometimes the *nominal par value.* The bond is said to be *redeemed* when the face value has been repaid and the bond surrendered to the debtor.

Bonds are usually redeemable *at par;* i.e. the sum named on the face of the bond is paid by the debtor, though sometimes, in order to secure an apparent reduction in the interest rate, they are made redeemable at a higher figure, or *above par.* Government bonds are usually redeemable at the option of the debtor after a certain date. For example, the law of 1870, under which the debt of the United States was refunded, provided for the issue of three kinds of bonds: one redeemable after 10 years and bearing 5% interest; one redeemable after 15 years and bearing $4\frac{1}{2}$% interest; and one redeemable after 15 years and bearing 4% interest.

Bonds are usually sold on the open market for what they will bring. If the selling price is the same as the face value, the bonds are said to be sold *at par;* if the selling price is greater than the par value, they are said to be sold *at a premium;* if it is less than the par value, they are said to be sold *at a discount.*

In determining the value of a bond, five things are taken into consideration: they are (1) the character of the security that

the debtor offers for the prompt payment of interest and principal as they fall due, (2) the price at which the bond is to be redeemed, (3) the rate of interest or dividend to be paid, (4) the rate of interest that is to be realized by the buyer, (5) the length of time to elapse before the bond will be redeemed.

The first of these elements cannot be measured mathematically, and so cannot be taken into consideration in any theoretical treatment of the problem of valuation. We have to consider only bonds concerning which, theoretically at least, there is no doubt as to the prompt and full payment of both interest and principal, and which bear a definite rate of interest, payable at definite dates.

It should be carefully noted that the rate of interest, or the dividend rate, named in the bond, has little or no relation to the current rate of interest, i.e. the rate to be realized by the investor. The dividend rate named in the bond is given merely for the sake of determining the periodical payment that is to be made to the investor. It is, however, an important element in the determination of the value of the bond. Neither the borrower nor the investor would be seriously affected by naming a higher or a lower dividend rate, for the price to be paid for the bond would be adjusted to meet the changed conditions.

PROBLEM. *To find the price of a bond purchased to yield a given rate of interest.*

Let C be the price to be paid on redemption, usually par value.

Let n be the number of years to elapse before redemption.

Let r be the ratio of the annual dividend to C, i.e. the *dividend rate*, or *cash rate.*

Let i be the rate to be realized — the *income rate*, or *investment rate.*

Let A_n be the purchase price n years before redemption.

The annual dividend will be rC, and if it is payable p times a year, the periodical dividend will be $\frac{rC}{p}$. The dividends constitute an annuity with annual rent rC for n years. Consequently, the value of the bond consists of two elements: viz. the present

value of the redemption price and the present value of the annuity constituted by the periodical dividend payments. The amount A_n which the investor can afford to pay is therefore the sum of the present values of these two elements. The present value of the redemption price is Cv^n, and the present value of the annuity constituted by the periodical dividend payments is $rCa_{\overline{n}|}^{(p)}$. Consequently,

$$A_n = Cv^n + rC \cdot a_{\overline{n}|}^{(p)}; \tag{1}$$

or, if the values of v^n and $a_{\overline{n}|}^{(p)}$ be written out in full,

$$A_n = C(1+i)^{-n} + \frac{rC}{p} \cdot \frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{p}}-1} \tag{2}$$

If interest is convertible m times a year,

$$1+i = \left(1+\frac{j}{m}\right)^m$$

by the formula for the relation between nominal and effective rates, and

$$A_n = C\left(1+\frac{j}{m}\right)^{-mn} + \frac{rC}{p} \cdot \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1}. \tag{3}$$

Finally, if the conversion interval for the interest coincides with the dividend period, $m = p$, and

$$A_n = C\left(1+\frac{j}{p}\right)^{-np} + \frac{rC}{p} \cdot \frac{1-\left(1+\frac{j}{p}\right)^{-np}}{\frac{j}{p}}, \tag{4}$$

or,

$$A_n = C\left[v^{np} + \frac{r}{p} \cdot a_{\overline{np}|}\right]\left(\text{at rate } \frac{j}{p}\right). \tag{5}$$

The purchase price, A_n, includes all expenses, such as brokerage and stamp taxes, incidental to the transfer of the bond from seller to buyer, for the bond is bought to yield a certain rate of

income to the buyer. If such expenses have been incurred, let the total amount be denoted by b. Then the amount realized by the seller is

$$A_n - b = Cv^n + rCa_{\overline{n}|}^{(p)} - b. \tag{6}$$

Formula (6) may be modified to correspond with any one of the formulas (2), (3), or (4), as occasion may require.

EXAMPLES

1. How much must be paid for a bond of \$100 with dividend rate at 6% nominal, payable January 1 and July 1 and redeemable after 20 years at par, to yield 5% nominal, convertible semiannually?

Solution. Since the conversion interval for the interest coincides with the dividend period, the solution will be given by formula (4). The present value of the redemption price will be

$$\$100 \times (1.025)^{-40} = \$37.243.$$

The total dividend paid in a year is \$6, and the semiannual dividend is \$3. The present value of the annuity constituted by the dividend payments is therefore

$$\begin{aligned}\$3 \times \frac{1-(1.025)^{-40}}{.025} &= \$3 \times 25.103 \\ &= \$75.309.\end{aligned}$$

It follows that the purchase price of the bond will be

$$\$37.243 + \$75.309 = \$112.55.$$

The premium to be paid is \$12.55.

2. How much must be paid for a bond of \$100 with dividend rate at 5% nominal, payable January 1 and July 1 and redeemable at par in 20 years, to yield 6% nominal, convertible semiannually?

49. The computation of the premium.

PROBLEM. *To find the premium that must be paid on a bond purchased to yield a given rate of interest.*

Of course the premium is known as soon as the purchase price is known. It turns out, however, that it is easier in most cases to compute the premium directly than to compute the purchase price, so that the best way to compute the purchase price is to compute the premium.

Let P denote the premium and C the par value. Then, by the notation of § 48,

$$A_n = C + P, \tag{1}$$

or,

$$P = A_n - C. \tag{2}$$

Replacing A_n by its value as given by (1) of § 48,

$$P = Cv^n + rCa_{\overline{n}|}^{(p)} - C \tag{3}$$

$$= C\left[\frac{r}{p}\frac{(1-v^n)}{[(1+i)^{\frac{1}{p}}-1]} - (1-v^n)\right],$$

or,

$$P = C \cdot \frac{(1-v^n)}{p[(1+i)^{\frac{1}{p}}-1]} \cdot \{r - p[(1+i)^{\frac{1}{p}}-1]\}. \tag{4}$$

The second factor in the last expression is exactly $a_{\overline{n}|}^{(p)}$, and by referring back to (3), § 23, we see that the expression $p[(1+i)^{\frac{1}{p}}-1]$ is the nominal rate, convertible p times a year, corresponding to the effective rate i. Let it be denoted by $j_{(p)}$, as in (3′), § 23. Using $a_{\overline{n}|}^{(p)}$ and $j_{(p)}$ instead of their equivalent expressions in (4), we obtain, finally,

$$\boldsymbol{P = C \cdot a_{\overline{n}|}^{(p)} \cdot (r - j_{(p)})}. \tag{5}$$

Formula (4) is readily adapted to logarithmic computation as soon as $a_{\overline{n}|}^{(p)}$ and $j_{(p)}$ are known.

The most important case is that in which the conversion interval for interest coincides with the dividend period. In this case $m = p$ and

$$a_{\overline{n}|}^{(p)} = \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]}$$

$$= \frac{1}{p}\frac{1-\left(1+\frac{j}{p}\right)^{-np}}{\frac{j}{p}}.$$

$$= \frac{1}{p}\left(a_{\overline{np}|} \text{ at rate } \frac{j}{p}\right). \quad \text{(See (5), § 32.)}$$

Moreover, when $m = p$, the expression

$$j_{(p)} = p\left[(1+i)^{\frac{1}{p}} - 1\right] = p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1\right]$$

reduces to j. Consequently, equation (4) reduces to

$$P = \frac{C(r-j)}{p}\left(a_{\overline{np}|} \text{ at rate } \frac{j}{p}\right). \tag{6}$$

Formulas (5) and (6) apply equally well to the computation of the discount when bonds are bought below par, the only difference being that for such bonds $r - j$ is negative. The discount is then to be considered as a sort of negative premium.

EXAMPLES

1. Find the purchase price of a bond for \$100 with dividend rate at 6% nominal, payable January 1 and July 1 and redeemable after 20 years at par, to yield 5% nominal, convertible semiannually.

Solution. We have $C = \$100$, $r = .06$, $j = .05$, $p = m = 2$, and $n = 20$. Hence, by formula (6),

$$\begin{aligned} P &= \frac{100\,(.06 - .05)}{2}\,(a_{\overline{40}|} \text{ at rate } .025) \\ &= .50 \times 25.103 \\ &= \$12.55. \end{aligned}$$

2. A bond for \$100 pays 6% nominal, payable January 1 and July 1, and is to be redeemed at par after 5 years. At what price must it be bought to yield 4% nominal, convertible January 1 and July 1?

3. Prove that after the kth interest payment has been made, the value of a bond redeemable at par after n years is given by the formula

$$A_{n-\frac{k}{p}} = C + C\frac{r-j}{p}\left(a_{\overline{np-k}|} \text{ at rate } \frac{j}{p}\right),$$

where C denotes the par value.

4. When bonds, redeemable on a certain date but due at a later date, are sold at a premium, their value is found on the supposition that they will be redeemed at the earlier date. Why?

Hint. The debtor can redeem the bond at par at any time after the earlier date. The question to be considered is whether it is worth more or less than par.

50. The amortization of the premium on a bond bought above par. When a bond which is to be redeemed at par is bought at a premium, some provision must be made for restoring to the original capital the amount of the premium; otherwise the capital will be impaired when the bond is redeemed. To illustrate, suppose the guardian of a minor buys a bond with par value \$1000 at a premium of \$500, intending to use the income for the maintenance of his ward. He has invested \$1500 belonging to his ward, and if the annual income from the bond has all been used for living expenses, *there will be only* \$1000 *to reinvest, instead of the original* \$1500, *when the bond is redeemed at par.* In other words, \$500 of the capital has disappeared as effectually as if it had been stolen. Indeed, the guardian should be held responsible for the disappearance of the \$500, just as though it had been stolen.

To remedy this difficulty, the premium should be considered as a debt to be repaid out of the surplus interest received. The gradual extinction of the premium through the application of the surplus interest is called the *amortization of the premium.* The value of the bond will diminish with each successive reduction of the premium until the date of redemption, when it should stand exactly at par. The value of a bond at any given date, when purchased to yield a given rate of interest, is termed the *book value.* An amortization schedule should be prepared exactly as in §§ 45, 46, and 47. Such a schedule should show the book value, the net income, and the amount of amortization at the end of every dividend period until the bond is redeemed.

ILLUSTRATIVE EXAMPLE. Consider the amortization of the premium on a bond for \$100, with interest at 6% (payable January 1 and July 1), and redeemable at par January 1, 1916, bought January 1, 1911, to yield 5% nominal, convertible semiannually.

Solution. By (6) of § 49 the premium is found to be \$4.376, and consequently the purchase price is \$104.376. At the end of each half year \$3 will be received, while 5% nominal on \$104.376 for a half year is only \$2.609; so that there is a surplus of \$0.391 available for amortization. Applying this surplus to the reduction of the premium, we see that the value of the bond is reduced

to \$103.985. For the second half year the net income will be $2\frac{1}{2}\%$ of \$103.985, or \$2.599, which will leave \$0.401 for amortization. The process is continued in this manner until, on January 1, 1916, the book value stands at \$100. The schedule may be arranged as follows:

Schedule of amortization of premium on 6% bond of the ———, redeemable January 1, 1916, January and July, bought January 1, 1911, to yield 5%:

Date	Total interest at 6%	Net income at 5%	For amortization	Book value
1911 Jan. 1 . . .	. . . . .	. . . . .	. . . . .	\$104.376
July 1 . . .	\$3.00	\$2.609	\$0.391	103.985
1912 Jan. 1 . . .	3.00	2.599	.401	103.584
July 1 . . .	3.00	2.589	.411	103.173
1913 Jan. 1 . . .	3.00	2.578	.422	102.753
July 1 . . .	3.00	2.568	.432	102.321
1914 Jan. 1 . . .	3.00	2.558	.442	101.879
July 1 . . .	3.00	2.547	.453	101.426
1915 Jan. 1 . . .	3.00	2.536	.464	100.962
July 1 . . .	3.00	2.524	.476	100.486
1916 Jan. 1 . . .	3.00	2.512	.488	100.000

EXAMPLE

Construct the schedule of amortization for a 6% bond with par value \$1000, payable in 10 years, and with dividend dates January 1 and July 1, bought January 1, 1912, to yield $4\frac{1}{2}\%$ nominal, convertible January 1 and July 1.

51. The accumulation of the discount on a bond bought below par. When the dividend rate is less than the specified income rate, the bond must be bought at a discount, and the value of the bond increases as the date of redemption approaches. At no time will the interest received on the bond be equal to the net income that should be received on the investment. The difference must be found by "writing up" the value of the bond, i.e. by adding the difference to the book value. The process of increasing the value of the bond at each dividend period by the difference between the required net income and the periodic installment of interest is called the *accumulation of the discount.*

A schedule of accumulation similar to the schedule of amortization of § 50 should be constructed when a bond is bought at a discount.

Illustrative Example. Consider the accumulation of the discount on a bond for $100, with interest at 4% nominal (payable January 1 and July 1), and redeemable January 1, 1916, bought January 1, 1911, to yield 5% nominal, convertible semiannually.

Solution. By (6) of § 49 the discount is found to be $4.376, so that the price of the bond is $95.624. At the end of every half year an interest payment of $2 is made, but for the first half year the net income at $2\frac{1}{2}$% per half year must be $2.391. In this case the interest received lacks $0.391 of meeting the required income, so that this amount must be "written on" to the value of the bond. The value upon which income at $2\frac{1}{2}$% per half year must be computed is, then, $96,015, and the required income at $2\frac{1}{2}$% per half year is $2.40, of which $0.40 must be written on to the value of the bond. With this explanation the construction of the schedule is easily understood.

Schedule of accumulation of discount on 4% *bond of the* ———, *payable January* 1, 1916, *January and July, bought January* 1, 1911, *to yield* 5% *:*

Date	Total interest at 4%	Income at 5%	Accumu-lation	Book value
1911 Jan. 1 . . .	. . . .	. . . . .	. . . .	$95.624
July 1 . . .	$2.00	$2.391	$.391	96.015
1912 Jan. 1 . . .	2.00	2.400	.400	96.415
July 1 . . .	2.00	2.410	.410	96.825
1913 Jan. 1 . . .	2.00	2.421	.421	97.246
July 1 . . .	2.00	2.431	.431	97.677
1914 Jan. 1 . . .	2.00	2.442	.442	98.119
July 1 . . .	2.00	2.453	.453	98.572
1915 Jan. 1 . . .	2.00	2.464	.464	99.036
July 1 . . .	2.00	2.476	.476	99.512
1916 Jan. 1 . . .	2.00	2.488	.488	100.000

EXAMPLE

Construct the schedule for the accumulation of the discount on a bond of $100, with interest at 5%, payable January 1 and July 1, redeemable January 1, 1917, purchased January 1, 1911, to yield 6% nominal, payable semiannually.

52. Schedule for the amortization of the premium in general terms when the interest interval and the conversion interval coincide. At the outset the value of the bond is

$$A_n = C + P = C + \frac{C(r-j)}{p} a_{\overline{np}|}, \tag{1}$$

where P is the premium and C the par value.

The interest received on the bond for one period is $\frac{rC}{p}$. The net income at nominal rate j for the first period is

$$A_n \cdot \frac{j}{p} = C\frac{j}{p} + \frac{C(r-j)}{p} a_{\overline{np}|} \cdot \frac{j}{p}$$

$$= C \cdot \frac{j}{p} + \frac{C(r-j)}{p}(1 - v^{np}). \tag{2}$$

The amortization for the first period, which we will denote by a_n, is the interest on the bond minus the net income, i.e.

$$a_n = \frac{rC}{p} - A_n \cdot \frac{j}{p}$$

$$= \frac{rC}{p} - \left[C \cdot \frac{j}{p} + \frac{C(r-j)}{p}(1 - v^{np})\right].$$

$$= \frac{C}{p}(r-j) v^{np}. \tag{3}$$

The book value at the end of the first period will be the value at the beginning less the amortization for the period, i.e.

$$A_{n-\frac{1}{p}} = A_n - a_n$$

$$= C + \frac{C(r-j)}{p} a_{\overline{np}|} - \frac{C(r-j)}{p} v^{np} \quad \text{(by (1) and (3))}$$

$$= C + \frac{C(r-j)}{p}(a_{\overline{np}|} - v^{np})$$

$$= C + \frac{C(r-j)}{p} a_{\overline{np-1}|}. \tag{4}$$

Consequently, the proper entry for each of the last three columns is found by subtracting 1 from np in each case. The last book value should of course be C. The schedule is as follows:

Schedule for the amortization of the premium on a bond with par value C and dividend rate r, bought to yield an income at nominal rate j, when the dividend period and the conversion interval coincide:

Beginning of period	Total interest at rate r	Net income at nominal rate j	Amortization	Book value
1	. . . .	. . . .	. . . . . .	$C+\frac{C(r-j)}{p}a_{\overline{np}\mid}$
2	$\frac{rC}{p}$	$C\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np})$	$\frac{C(r-j)}{p}v^{np}$	$C+\frac{C(r-j)}{p}a_{\overline{np-1}\mid}$
3	$\frac{rC}{p}$	$C\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np-1})$	$\frac{C(r-j)}{p}v^{np-1}$	$C+\frac{C(r-j)}{p}a_{\overline{np-2}\mid}$
4	$\frac{rC}{p}$	$C\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np-2})$	$\frac{C(r-j)}{p}v^{np-2}$	$C+\frac{C(r-j)}{p}a_{\overline{np-3}\mid}$
...	...	...	...	...
$np+1$	$\frac{rC}{p}$	$C\frac{j}{p}+\frac{C(r-j)}{p}(1-v)$	$\frac{C(r-j)}{p}v$	C

EXAMPLES

1. Prove that the net interests for the first, second, third, $\cdots$, periods are

$$A_{n}\cdot\frac{j}{p}=C\cdot\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np}),$$
$$A_{n-\frac{1}{p}}\cdot\frac{j}{p}=C\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np-1}),$$
$$A_{n-\frac{2}{p}}\cdot\frac{j}{p}=C\frac{j}{p}+\frac{C(r-j)}{p}(1-v^{np-2}).$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

2. Prove that the amortizations for the successive periods are,

for the first, $\frac{C(r-j)}{p}v^{np}$;

for the second, $\frac{C(r-j)}{p}v^{np-1}$;

for the third, $\frac{C(r-j)}{p}v^{np-2}$, etc.

3. Construct the schedule for the accumulation of the discount in general terms.

53. Bonds bought between two interest dates. When a bond is bought at some time between two interest dates, the purchase price will be, theoretically, the value at the last interest date, accumulated to the date of purchase, or the sum of the value at the next interest date and the interest for the period, discounted to the date of purchase. In actual business transactions the procedure is somewhat different.

PROBLEM. *To find the theoretical value of a bond bought $\frac{1}{k}$th of a period after an interest payment has been made.*

Let n denote the time from the last interest date to the date of redemption. If interest on the bond is paid p times a year, the time from the date of purchase to the date of redemption is $n-\frac{1}{kp}$. Let A_n be the value immediately after the last interest payment was made, and $A_{n-\frac{1}{kp}}$ the value on the date of purchase. Then the value $A_{n-\frac{1}{kp}}$ will be the value A_n accumulated for the time $\frac{1}{kp}$; i.e.

$$A_{n-\frac{1}{kp}}=A_n(1+i)^{\frac{1}{kp}}=A_n\left(1+\frac{j}{m}\right)^{\frac{m}{kp}}, \tag{1}$$

where A_n is determined by the formula (1), § 48. If the interest period coincides with the conversion interval, $m=p$, and

$$A_{n-\frac{1}{kp}}=A_n\left(1+\frac{j}{p}\right)^{\frac{1}{k}}. \tag{2}$$

In practice the amortization for any fractional part of a period is assumed to be proportional to the time, and the buyer pays to the seller the simple interest that has accrued on the bond at rate j since the last interest date. The value in practice is therefore the value at the last interest date, plus one kth part of the interest on the bond for a period, minus one kth part of the amortization for the particular period within which the purchase is made. This method is exactly equivalent to accumulating the value at the last interest date by *simple* instead of by compound interest (see Example 3, below).

PROBLEM. *To find the value in practice of a bond bought one kth part of a period after an interest payment has been made.*

Let A_n and $A'_{n-\frac{1}{kp}}$ be the values at the last interest date and at the date of purchase. Then, since the value in practice is A_n accumulated for one kth part of a year at simple interest,

$$A'_{n-\frac{1}{kp}} = A_n\left(1+\frac{j}{kp}\right). \tag{3}$$

EXAMPLES

1. Find the value of a bond for $100, with interest at 6% nominal (payable January 1 and July 1), and redeemable January 1, 1916, bought March 1 1911, to yield 5% nominal, convertible semiannually.

Solution. By the formulas of § 49 the value on January 1, 1911, was $104.376. The time since the last interest payment may be taken as $\frac{1}{3}$ of a period, or $\frac{1}{6}$ of a year. Formula (2) gives, when $A_n = \$104.376$,

$$\begin{aligned} A_{5-\frac{1}{6}} &= \$104.376\,(1.025)^{\frac{1}{3}} \\ &= \$105.239. \end{aligned}$$

This is the theoretical value.

To find the value in practice, we have, by formula (3),

$$\begin{aligned} A'_{5-\frac{1}{6}} &= \$104.376 \times \left(1+\frac{.05}{6}\right) \\ &= \$105.246. \end{aligned}$$

The result shows, as we might know from theoretical considerations, that the method in practice is slightly to the buyer's disadvantage.

2. Construct the schedule for the amortization of the premium on a 6% bond for $100, interest payable January 1 and July 1, redeemable January 1, 1921, purchased October 1, 1911, to yield 5%, convertible semiannually.

3. Prove the statement made above, that the value of a bond purchased one kth part of a period after an interest date, defined to be the value at the last interest date, plus one kth part of the interest on the bond for the period, minus one kth part of the amortization for the particular period within which the purchase is made, is equivalent to the value at the last interest date accumulated at simple interest from the last interest date to the date of purchase.

4. Show that the theoretical value of a bond bought between two interest dates will be the same whether the value at the last interest date be accumulated to the date of purchase or the value at the next interest date be discounted to the date of purchase.

MISCELLANEOUS EXAMPLES AND PROBLEMS

1. Find the purchase price of 5% bonds, interest payable quarterly and redeemable at par in 10 years, to yield 6% effective.

2. Find the purchase price of 6% bonds, interest payable quarterly, redeemable at par in 10 years, to yield the purchaser 5% effective on the investment.

3. Find the purchase price of 4% bonds, interest payable half yearly and redeemable in 20 years at 120, to yield 5% effective to the purchaser.

4. A man owns some bonds paying 6% nominal, payable quarterly and having 8 years to run, when they will be redeemed at 110. He pays a commission man $\frac{3}{4}$%, computed on the face value, for selling them, and the government imposes a stamp tax of 2 cents on every hundred dollars of par value. What is the selling price if the purchaser is to realize 6% on his investment?

5. If 5% bonds are redeemable after 20 years at 120, at the option of the debtor, and money can be borrowed at $4\frac{1}{2}$% at the end of the period, will the debtor be likely to redeem the bonds or not? Explain why.

6. Find the expression for the purchase price of a bond by accumulating the interest to the date of maturity of the bond and finding the present value of the sum that would be due at maturity. Show that the result agrees with the result obtained in formula (2) of § 48.

7. One thousand bonds bearing interest at 5% nominal, payable semi-annually, are issued in denomination of $1000 each. At the end of 3 years, and at the end of each year thereafter up to and including the twentieth, such a number of bonds is to be redeemed at 105 that the whole issue shall be paid in 20 years. How many bonds must be redeemed each year in order that the sum of the interest charge on the unpaid bonds and the amount paid to redeem bonds shall be the same for each of the 18 years? (See *Engineering News*, October 30, 1902.)

54. Installment bonds. An *installment bond* is an interest-bearing bond payable, principal and interest, in equal annual installments. The problem of determining the annual installment to be paid on such a bond is therefore precisely the problem of § 42, where the annual payment required to extinguish a debt K by a series of equal annual installments is found.

Let C be the face value of the bond, n the time to elapse before the bond is paid, r the rate of interest named in the bond, and R the annual installment. By (1), § 42, the annual installment is

$$R = C\left(\frac{1}{a_{\overline{n}|}} \text{ at rate } r\right). \tag{1}$$

PROBLEM. *To find the price at which an installment bond must be purchased to yield a given rate on the investment.*

Let A be the purchase price and i the rate of interest to the investor. The purchaser is buying an annuity of R per annum, which must yield interest at rate i. The purchase price A is therefore the present value of an annuity of R per annum with interest at rate i, and the value of A is given by the formula

$$A = R(a_{\overline{n}|} \text{ at rate } i). \tag{2}$$

If R be replaced in (2) by its value as given by (1), the result is

$$A = C\frac{(a_{\overline{n}|} \text{ at rate } i)}{(a_{\overline{n}|} \text{ at rate } r)}, \tag{3}$$

or, written out in full,

$$A = C \cdot \frac{r}{i}\frac{1-(1+i)^{-n}}{1-(1+r)^{-n}}. \tag{4}$$

EXAMPLES

1. What is the value of a 4% installment bond with face value \$1000, to be paid in 10 annual installments, in order that the purchaser may receive 3% on his investment.

Solution. By formula (3) the solution would be, using five-place tables,

$$\begin{aligned} A &= \$1000\frac{a_{\overline{10}|} \text{ at rate } .03}{a_{\overline{10}|} \text{ at rate } .04} \\ &= \$1000\frac{8.53020}{8.11090} \\ &= \$1051.70. \end{aligned}$$

The solution could have been obtained by finding $\frac{1}{a_{\overline{10}|}}$ at rate .04 from the tables, and thus avoiding the division. In this way

$$\begin{aligned} A &= \$1000 \times 8.53020 \times .12329 \\ &= \$1051.69. \end{aligned}$$

If six-place tables were used, the results would agree exactly.

2. For what price must a 5% installment bond for \$100, to be paid in 10 years, be purchased to yield 6% to the purchaser?

3. Find the formula for the purchase price of an installment bond payable quarterly for n years, with interest at rate r, to yield interest at rate i to the purchaser.

4. Prove formulas (3) and (4) by accumulating the purchase price and setting the result equal to the amount of the annuity.

CHAPTER IX

SINKING FUNDS AND DEPRECIATION

55. Sinking funds. A *sinking fund* is a fund created for the purpose of paying a debt when the debt falls due. The sums set aside for sinking funds are supposed to be productively invested. If the same amount is set aside at regular intervals, the amounts form an annuity for which either the amount, the present value, or the annual rent may be computed when the requisite data are given.

PROBLEM. *To find the amount that must be set aside annually to extinguish a given debt K due in n years.*

Suppose the debt K falls due in n years, and that sums paid to the sinking fund are invested at rate i. The problem is, then, to determine the annual rent R when the amount K, the time n, and the rate i are known. This is exactly the problem of § 36. We shall have

$$Rs_{\overline{n}|} = K;$$

whence

$$R = K \cdot \frac{1}{s_{\overline{n}|}}. \tag{1}$$

This equation is identical with equation (4) of § 36, which was called the *sinking-fund equation.* The value of $\frac{1}{s_{\overline{n}|}}$ may be found from a table of values of "the annuity that will amount to 1," or, if such a table is not at hand, it may be found from a table of values of "the annuity that 1 will purchase" by means of the fundamental relation

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i. \qquad \text{(See (2), § 38.)}$$

ILLUSTRATIVE EXAMPLE. A city incurs a debt of $100,000, agreeing to pay interest promptly and to establish a sinking fund to repay the principal at the end of 20 years. If the payments to the sinking fund can be invested at 4% effective, what will be the annual payment to the fund?

Solution. By formula (1),

$$\begin{aligned} R &= \$100{,}000\left(\frac{1}{s_{\overline{20}|}} \text{ at } 4\%\right) \\ &= \$100{,}000\left(\frac{1}{a_{\overline{20}|}} - .04\right) \\ &= \$3358.175. \end{aligned}$$

PROBLEM. *To find the amount that must be set aside annually to pay interest at rate i on a debt K and create a sinking fund that will extinguish the debt in n years, when the rate of accumulation differs from i.*

Let i' be the rate at which the sinking fund can be accumulated, and let R' be the amount that must be set aside. It will require Ki to pay the interest as it falls due each year. The payment into the sinking fund will be, by (1),

$$R = K \cdot \left(\frac{1}{s_{\overline{n}|}} \text{ at rate } i'\right).$$

The required amount is, therefore,

$$\boldsymbol{R' = Ki + K \cdot \left(\frac{1}{s_{\overline{n}|}} \text{ at rate } i'\right).} \tag{2}$$

If $\frac{1}{s_{\overline{n}|}}$ be replaced by its value $\frac{1}{a_{\overline{n}|}} - i'$, according to § 38,

$$R' = Ki + K\left(\frac{1}{a_{\overline{n}|}} - i'\right),$$

or,

$$\boldsymbol{R' = K(i - i') + K\left(\frac{1}{a_{\overline{n}|}} \text{ at rate } i'\right).} \tag{3}$$

COROLLARY. If $i' = i$, the first term of (3) disappears and the formula is simply

$$R' = K\frac{1}{a_{\overline{n}|}}. \tag{4}$$

✓

But formula (4) is precisely the formula (1) of § 42 for the annual payment that will extinguish an interest-bearing debt in equal annual installments. This important fact may be stated as follows:

When the sinking fund can be accumulated at the rate of interest borne by the debt, the amount required each year for interest and for the sinking-fund payment is exactly equal to the amount that would be required to pay the debt, principal and interest, in the same number of equal annual installments.

PROBLEM. *To find the amount in the sinking fund at any given time, when the dividend rate and the accumulation rate are equal.*

Let n be the time that has elapsed since the sinking fund was established, and let S_{n_1} be the amount in the sinking fund at the expiration of n_1 years. The annual payment into the sinking fund is, by (1), $K \cdot \frac{1}{s_{\overline{n}|}}$. This sum is the annual rent of the annuity whose amount, S_{n_1}, we wish to find. By the formula for the amount of an annuity whose annual rent is known (see (6), § 31), the amount is

$$S_{n_1} = K \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{n_1}|}, \tag{5}$$

or

$$S_{n_1} = K \frac{(1+i)^{n_1} - 1}{(1+i)^n - 1}. \tag{5'}$$

Either (5) or (5′) is well adapted to logarithmic computation, since $s_{\overline{n}|}$, $s_{\overline{n_1}|}$, $(1+i)^n$, and $(1+i)^{n_1}$ may be found directly from the tables.

Sometimes when a bond issue has been sold, sinking-fund payments are used to purchase a part of the identical bonds for the redemption of which the sinking fund was established.* When sinking funds are so invested, the bonds thus purchased may be "kept alive" for interest purposes or they may be cancelled. In case the bonds are kept alive, the company simply transfers to its sinking-fund account on each interest date the

* H. R. Hatfield, Modern Accounting. D. Appleton and Company, 1909.

interest that would be due on the bonds in hand. From the mathematical point of view this method of investing the sinking fund does not differ from the ordinary method, except that the rate of interest realized will probably be somewhat higher than the savings-bank rate. It is of course immaterial whether a company receives interest on its own bonds or on the bonds of another corporation.

If, however, the bonds are retired, that portion of the debt which the retired bonds represent is paid and ceases to bear interest. There is in such cases an interesting problem which is formulated as follows:

PROBLEM. *To determine the annual payment that will be required to pay interest on a bonded debt, and to purchase for retirement a sufficient number of bonds so that the bonds will all be retired at the end of n years.*

Let K denote the amount of the debt, i the effective rate of interest paid by the bonds, $1+k$ the price the bonds will bring in the open market, and x the required annual payment.

At the end of the first year the accrued interest will be Ki, and after this is paid, the amount available for the purchase of bonds will be $x-Ki$; the face value of the bonds purchased at the price $1+k$ will be $\dfrac{x-Ki}{1+k}$; and the amount of the debt remaining unpaid will be $K-\dfrac{x-Ki}{1+k}$. Similarly, at the end of the second year we shall have

interest due $= Ki - \dfrac{x-Ki}{1+k}i$;

amount available for purchasing bonds $= x - Ki + \dfrac{x-Ki}{1+k}i$;

par value of bonds purchased $= \dfrac{x-Ki}{1+k} + \dfrac{x-Ki}{(1+k)^2}i$;

debt outstanding $= K - 2\dfrac{x-Ki}{1+k} - \dfrac{x-Ki}{(1+k)^2}i$.

If the process be carried a step further, the law by which the various quantities may be computed will begin to show itself. At the end of the third year the situation will be as follows:

$$\text{interest payment} = Ki - 2\frac{x-Ki}{1+k}i - \frac{x-Ki}{(1+k)^2}i^2;$$

$$\text{amount paid for bonds} = x - Ki + 2\frac{x-Ki}{1+k}i + \frac{x-Ki}{(1+k)^2}i^2;$$

$$\text{par value of bonds purchased} = \frac{x-Ki}{1+k} + 2\frac{x-Ki}{(1+k)^2}i + \frac{x-Ki}{(1+k)^3}i^2;$$

$$\begin{aligned}\text{debt outstanding} &= K - 2\frac{x-Ki}{(1+k)} - \frac{x-Ki}{(1+k)^2}i \\ &\quad - \left[\frac{x-Ki}{(1+k)} + 2\frac{x-Ki}{(1+k)^2} + \frac{x-Ki}{(1+k)^3}i^2\right] \\ &= K - 3\frac{x-Ki}{1+k} - 3\frac{x-Ki}{(1+k)^2}i - \frac{x-Ki}{(1+k)^2}i^2.\end{aligned}$$

If we take out of the first and second of these expressions, as far as is possible, the factor $x - Ki$, out of the third the factor $\frac{x-Ki}{1+k}$, and out of the fourth the factor $\frac{x-Ki}{i}$, and compare the remaining factors with the binomial expansions for powers of $1 + \frac{i}{1+k}$, we obtain the following expressions:

$$\text{interest payment} = Ki - (x - Ki)\left[\left(1 + \frac{i}{1+k}\right)^2 - 1\right];$$

$$\text{amount paid for bonds} = (x - Ki)\left(1 + \frac{i}{1+k}\right)^2;$$

$$\text{par value of bonds purchased} = \frac{x-Ki}{1+k}\left(1 + \frac{i}{1+k}\right)^2;$$

$$\text{debt outstanding} = K - \frac{x-Ki}{i}\left[\left(1 + \frac{i}{1+k}\right)^3 - 1\right].$$

These forms lead us to believe that the expressions for these amounts at the end of n years would be

$$\text{interest payment} = Ki - (x - Ki)\left[\left(1 + \frac{i}{1+k}\right)^{n-1} - 1\right]; \tag{6}$$

$$\text{amount paid for bonds} = (x - Ki)\left(1 + \frac{i}{1+k}\right)^{n-1}; \tag{7}$$

$$\text{par value of bonds purchased} = \frac{x - Ki}{1+k}\left(1 + \frac{i}{1+k}\right)^{n-1}; \tag{8}$$

$$\text{debt outstanding} = K - \frac{x - Ki}{i}\left[\left(1 + \frac{i}{1+k}\right)^{n} - 1\right]. \tag{9}$$

That these results are correct is shown by mathematical induction, for if we assume them to be true for $n = r$, an easy computation shows that they are true for $n = r + 1$, and consequently for all values of n.

But if the debt is to be paid in n years, the debt outstanding at the end of n years must be zero; i.e.

$$K - \frac{x - Ki}{i}\left[\left(1 + \frac{i}{1+k}\right)^{n} - 1\right] = 0.$$

Transposing and dividing by the coefficient of $x - Ki$, we find

$$x - Ki = \frac{Ki}{\left(1 + \frac{i}{1+k}\right)^{n} - 1};$$

whence

$$x = Ki\left[1 + \frac{1}{\left(1 + \frac{i}{1+k}\right)^{n} - 1}\right], \tag{10}$$

or,

$$x = Ki\frac{(1+i+k)^{n}}{(1+i+k)^{n} - (1+k)^{n}}. \tag{11}$$

Of course no actual business transaction could be carried out in exact accordance with formula (11), for in the first place an integral number of bonds must be retired each year, and in the second place the assumed market value of the bonds, $1 + k$,

would probably vary from year to year. However (and this is the important point), *the value of x given by the formula will furnish a starting point by means of which a schedule showing approximately the process of retirement may be constructed.*

ILLUSTRATIVE EXAMPLE. A city issues bonds for \$200,000, to pay for a high-school building. The bonds are of denomination \$1000 and bear interest at 5% effective. If they can be repurchased at 98, what amount must be set aside annually to pay interest and buy back bonds so that the debt will all be paid in 20 years?

Solution. For this problem $K = \$200,000$, $i = .05$, $k = -.02$, $n = 20$. We have, then,

$$\begin{aligned} x &= \$200,000 \times .05 \frac{(1.03)^{20}}{(1.03)^{20} - (.98)^{20}} \\ &= \frac{18,061.112}{1.13851} \\ &= \$15,864, \text{ nearly.} \end{aligned}$$

For reasons noted above it would be useless to carry the division further. After the interest, \$10,000, is paid, there would be approximately \$5864 available for the purchase of bonds at 98. The par value of the bonds purchased would be a little less than \$6000, so that 6 bonds of face value \$1000 or 60 bonds of face value \$100 could be purchased. The following year the interest charge would be \$4700, and the same number of bonds could be retired. The process furnishes little more than the means of making a rough estimate as to what would happen.

EXAMPLES

1. A debt of \$200,000 is to be paid in 20 years by means of a sinking fund established at the time the debt is incurred. If the sinking fund can be accumulated at 4% effective, how large will the fund be at the end of 10 years?

Solution. By formula (5), the amount in the fund will be

$$\begin{aligned} S_{10} &= \$200,000 \cdot \frac{1}{s_{\overline{20}|}} \cdot s_{\overline{10}|} \\ &= \$200,000 \times .03358175 \times 12.006107 \\ &= \$80,637.216. \end{aligned}$$

Or, by formula (5′), $S_{10} = \$200,000 \cdot \dfrac{(1.04)^{10} - 1}{(1.04)^{20} - 1}$

$$\begin{aligned} &= \$200,000 \cdot \frac{.48024428}{1.19112314} \\ &= \$80,637.216. \end{aligned}$$

✓ **2.** What will be the annual payment into a sinking fund to be accumulated at 4% effective, established for the purpose of extinguishing a debt of \$225,000, due in 20 years?

3. If a sinking fund can be accumulated at 4% nominal, convertible semiannually, what must be the semiannual payment to a sinking fund which will provide for the extinction of the debt at the end of 20 years?

4. A man gives a mortgage for $4000 on his home, paying interest at the rate of 6%, payable annually, and depositing a certain sum each year in a savings bank which pays 4% nominal, convertible semiannually. What sum must he set aside annually to pay interest and to pay off the mortgage in 10 years?

5. A city incurs a debt of $200,000. Which would be better, to pay the debt, principal and interest at $6\frac{1}{4}$%, in 20 equal annual installments, or to pay 6% interest each year on the debt and pay a fixed amount annually into a sinking fund to be accumulated for 20 years at 4%?

6. A debt of $10,000, due in 10 years and bearing interest at the rate of 6%, payable annually, can be paid by means of a sinking fund which can be accumulated at 4% effective. What is the rate if the debtor can arrange to pay principal and interest in 10 equal annual installments, in order to make the terms equivalent to the sinking-fund plan?

7. A city issues twenty-year bonds of denomination $100, bearing interest at 5% nominal, payable semiannually, to the amount of $200,000. A sinking fund can be accumulated at 4% effective. Which would be better, to accumulate a sinking fund, to which equal annual payments shall be made, or to buy back bonds in such manner that the annual payments for interest and for bonds repurchased will be as nearly equal as possible throughout the twenty-year period.

56. Depreciation. *Depreciation* has been defined as "loss in the value of physical property due to use, which cannot be made good by current repairs." No matter how much care is taken to keep machinery or buildings in good repair, there will come a time when a new machine or new buildings must be put in place of the old. Since this is true of practically every kind of physical property that is used in the conduct of business of any kind whatsoever, common sense demands that the business man should make some definite provision for the replacement of property that will be destroyed through use. To neglect such provision is to invite certain disaster in the not far distant future.

To illustrate, suppose a man pays $1000 cash for an auto truck, and engages in the business of hauling freight. He earns, say $10 a day, out of which he pays for fuel and oil, and for any repairs that may be necessary. The remainder he spends for

house rent, for food, for clothing, for amusements, or for any other purpose. At the end of a period varying from five to ten years his machine is worn out beyond all possibility of repair, and, having no money with which to purchase a new machine, he must go out of business. The case is not essentially different if the machine is bought on time.

The trouble might have been averted if this man had set aside each year a definite sum which would have accumulated to an amount sufficient to purchase a new machine by the time the old one was worn out. A fund set aside to replace a piece of depreciable property when it is worn out is called a *depreciation fund.*

A depreciation fund is essentially a sinking fund, as may be seen clearly if we think of the cost of the depreciable property as a debt to be repaid when the property is cast aside. For the sake of brevity the depreciable property will be called an *asset.* A piece of depreciable property usually possesses a *residual,* or *scrap, value* when it is thrown aside as worn out. The scrap value of a steam engine is nearly equal to the value of an equivalent weight of old iron. Copper wire used in electrical work is worth several cents per pound after the insulation becomes so worn that the wire cannot be used safely for electrical purposes. The cost new of an asset, less its scrap value, is called the *wearing value,* or the *depreciable value.* The length of time to elapse before the value of an asset is reduced to its scrap value by reason of age or wear is called its *probable life,* or simply its *life.* The determination of the probable life is a matter of observation and must be made by the engineer or the architect, or by some person familiar with the construction and use of the asset.

The most convenient way to accumulate a depreciation fund is to set aside a stated amount each year for the purpose. Whether this amount is to be invested in securities or placed in a savings bank or put back into the business is a question that need not be discussed here. It is important to note, however, that, for small concerns at least, such funds could be made to earn a low rate of interest by depositing them in a savings

bank, so that *we are bound to consider them as accumulated at a rate at least as high as the savings-bank rate.*

In all discussions of depreciation the fundamental principle to be considered is that *capital invested in productive enterprises must not be impaired.*

PROBLEM. *To find the amount that must be set aside annually for a depreciation fund to provide for the replacing of an asset worn out by use.*

Let C be the cost new, W the wearing value, n the probable life, S the scrap value, i the effective rate at which the fund can be accumulated, and D the annual payment to the depreciation fund. The value to be replaced is

$$W = C - S. \tag{1}$$

The wearing value, W, may therefore be looked upon as the debt to be repaid at the expiration of n years. By formula (1) of § 55 we find

$$D = W \cdot \frac{1}{s_{\overline{n}|}}. \tag{2}$$

ILLUSTRATIVE EXAMPLE. What sum must be set aside annually to provide for replacing an auto truck costing \$1000 and having a probable life of 5 years, if the depreciation fund can be accumulated at 4% effective, and the scrap value is \$25?

Solution. The wearing value, W, is \$975, so that, by formula (2) above,

$$\begin{aligned} D &= \$975 \times \frac{1}{s_{\overline{5}|}} \\ &= \$975 \times \left(\frac{1}{a_{\overline{5}|}} - .04\right) \\ &= \$975 \times .184627 \\ &= \$180.01. \end{aligned}$$

PROBLEM. *To find the amount in the depreciation fund at the end of n_1 years.*

This is precisely equivalent to the third problem in § 55. Let D_{n_1} denote the amount in the depreciation fund at the end of

n_1 years. Since the annual rent of the annuity has been found by formula (2) to be $W \cdot \frac{1}{s_{\overline{n}|}}$, the amount at the end of n_1 years will be

$$D_{n_1} = W \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{n_1}|} = W \frac{(1+i)^{n_1}-1}{(1+i)^n-1}. \tag{3}$$

For example, in the problem of the auto truck above, the amount of the annual payment was found to be

$$D = W \cdot \frac{1}{s_{\overline{5}|}} = \$180.01.$$

The amount of an annuity of \$180.01 for 4 years, when accumulated at 4%, would be

$$D_4 = \$180.01 \times s_{\overline{4}|}(\text{at } 4\%) = \$180.01 \times 4.246464$$
$$= \$764.406.$$

PROBLEM. *To find the annual payment to the depreciation fund when the depreciable property consists of several kinds of assets having probable lives of various lengths.*

Let W_1, W_2, W_3, $\cdots$, be the wearing values of the various parts of a "plant," and n_1, n_2, n_3, $\cdots$, their probable lives. The total charge for depreciation will be the sum of the charges for the separate parts. The separate charges will be, by formula (2),

$$W_1 \cdot \frac{1}{s_{\overline{n_1}|}}, \quad W_2 \cdot \frac{1}{s_{\overline{n_2}|}}, \quad W_3 \cdot \frac{1}{s_{\overline{n_3}|}}, \cdots,$$

and consequently the total annual charge will be

$$D = W_1 \cdot \frac{1}{s_{\overline{n_1}|}} + W_2 \cdot \frac{1}{s_{\overline{n_2}|}} + W_3 \cdot \frac{1}{s_{\overline{n_3}|}} + \cdots. \tag{4}$$

Suppose, for example, we wish to find the annual charge for depreciation for an electrical power plant which consists of four parts: viz. a building with wearing value \$10,000 and life 50 years, an engine with wearing value \$5000 and life 25 years, a boiler with wearing value \$2000 and life 15 years, a dynamo with wearing value \$6000 and life 18 years, when money can be accumulated at 4%.

By formula (4) the total charge for depreciation will be

$$D = \$10{,}000 \cdot \frac{1}{s_{\overline{50|}}} + \$5000 \cdot \frac{1}{s_{\overline{25|}}} + \$2000 \cdot \frac{1}{s_{\overline{15|}}} + \$6000 \cdot \frac{1}{s_{\overline{18|}}}$$

$$= \$65.502 + \$120.060 + \$99.882 + \$233.960$$

$$= \$519.404.$$

These results can be put into tabular form as follows:

Part	Life of part	Wearing value	Annual depreciation charge
Building . .	50 years	$10,000	$65.502
Engine . . .	25 years	5,000	120.060
Boiler . . .	15 years	2,000	99.882
Dynamo . .	18 years	6,000	233.960
		$23,000	$519.404

If the probable life has been correctly given, it is clear that the renewal of each part of the plant will be provided for when the part must be discarded as worn out, since the depreciation on each part has been figured separately.

EXAMPLES

1. On a 4% basis, find the annual charge for depreciation for a plant consisting of four parts, with costs, scrap values, and probable lives as given in the following table:

Part	Cost new	Scrap value	Life
A	$100,000	$5000	50 years
B	25,000	3000	25 years
C	20,000	1000	15 years
D	30,000	2500	20 years

2. A building costs new $200,000. Small repairs cost $1000 per annum, and every 10 years the building must be thoroughly renovated at a cost of $20,000. If the building must be torn down in 50 years, with a salvage value of $5000, what sum must be set aside annually to cover the cost of repairs, renovation, and depreciation?

3. The rules of the Interstate Commerce Commission (1909) require that a monthly charge for depreciation shall be allowed toward operating expenses for all steam and electric locomotives. What would be the monthly depreciation charge for a steam locomotive costing, new, \$12,000, having a probable life of 25 years and a scrap value of \$1000, provided the monthly charge is taken to be one twelfth of the annual depreciation charge and the rate of interest is 3% effective? What would be the monthly charge computed by the formula for an annuity payable twelve times a year?

4. In a recent investigation of a public utility the values of the various parts were found to be as follows:

Part	Life	Wearing value
A	10 years	\$3,157,930
B	25 years	3,076,239
C	40 years	2,677,425
D	15 years	674,867
E	12 years	666,176
F	12 years	1,036,062
G	30 years	146,059
H	10 years	2,862,091
I	10 years	679,206
J	50 years	1,648,378
K	5 years	134,828

What should be the annual charge for depreciation on a 3% basis?
—W. J. Hagenah, *Chicago Telephone Company Investigation.*

57. The composite life of a plant. The notion of the life of a plant as a whole is of considerable importance in several connections. It has no meaning, however, until it is defined. The *composite life* of a plant is defined as the time in which the total annual depreciation charge for the plant would amount to the total wearing value when accumulated at an assumed rate of interest.

PROBLEM. *To determine the composite life of a plant composed of several parts having different probable lives.*

According to the definition the composite life would be the value of n determined from the equation

$$\text{Total wearing value} = s_{\overline{n|}} \times \text{total annual depreciation charge.}$$

Let W denote the total wearing value, i.e. the sum of the wearing values of the several parts, and let D be the total annual depreciation charge. Then, by definition,

$$W = s_{\overline{n|}} D. \tag{1}$$

When $s_{\overline{n|}}$ is replaced by its value, viz. $\dfrac{(1+i)^n - 1}{i}$, equation (1) becomes

$$W = \frac{(1+i)^n - 1}{i} D, \tag{2}$$

which in turn reduces easily to the form

$$(1+i)^n = 1 + \frac{W}{D} i.$$

Solving this exponential equation for n, we find

$$n = \frac{\log\left(1 + \frac{W}{D} i\right)}{\log (1 + i)}. \tag{3}$$

The ratio of the total annual depreciation charge D to the total wearing value W may be called the *rate of depreciation.* Denoting the rate of depreciation by d, we have, by definition,

$$d = \frac{D}{W} = \frac{1}{s_{\overline{n|}}}. \tag{4}$$

If the value of the ratio $\dfrac{D}{W}$ be replaced by d in equation (3), that formula reduces to

$$n = \frac{\log\left(1 + \frac{i}{d}\right)}{\log (1 + i)}. \tag{5}$$

Illustrative Example. Suppose we wish to determine the composite life of the electric power plant discussed in § 56.

Solution. On a 4% basis the annual depreciation charge for the building with life 50 years and wearing value \$10,000 was found to be \$65.502; for the engine with life 25 years and wearing value \$5000, \$120.060; for the boiler with life 15 years and wearing value \$2000, \$99.882; for the dynamo with life 18 years and wearing value \$6000, \$233.960. The total wearing value W is \$23,000, and

the total annual charge for depreciation is \$519.404. Consequently, the rate of depreciation is

$$d = \frac{519.404}{23{,}000} = .022583.$$

By formula (5),

$$n = \frac{\log\left(1 + \frac{.04}{.022583}\right)}{\log(1.04)}$$

$$= \frac{\log 2.7712}{\log(1.04)}$$

$$= \frac{0.44267}{0.01703}$$

$$= 25.994 \text{ years.}$$

It is easy to get an approximate solution directly from the tables, for from formula (1) we have

$$\frac{1}{s_{\overline{n}|}} = \frac{D}{W} = d. \tag{6}$$

But

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i,$$

so that

$$\frac{1}{a_{\overline{n}|}} = d + i. \tag{7}$$

In the present problem

$$\frac{1}{a_{\overline{n}|}} = .022583 + .04$$

$$= .062583.$$

From the table for "the annuity that 1 will purchase," i.e. $\frac{1}{a_{\overline{n}|}}$, we see that .062583 is very nearly the value corresponding to $n = 26$.

The notion of composite life is important when considered in connection with bonds secured by the property of a plant. For example, in the case just considered, bonds secured by the property of the plant should not be issued for a time exceeding the composite life of the plant, viz. twenty-six years. Indeed, to provide a proper margin of safety, bonds should not be issued for a time exceeding twenty years.

58. The wearing value of a single asset whose probable life is given.

PROBLEM. *To find the wearing value, at a given time, of an asset whose probable life is known.*

By virtue of the principle that capital invested should not be impaired, the value of an asset at any time, plus the amount in the depreciation fund for the redemption of the cost of the asset, should be equal to the wearing value new.

Let W denote the wearing value when new, and W_t the wearing value at the end of t years. The annual depreciation charge is, by (2) of § 56, $W \cdot \frac{1}{s_{\overline{n}|}}$, and at the end of t years the accumulation in the depreciation fund will be $W \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{t}|}$, so that

$$W_t = W - W \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{t}|}. \qquad (1)$$

Scrap value = Cost − accrued Dep.

Replacing $s_{\overline{n}|}$ and $s_{\overline{t}|}$ by their values, and taking out the common factor W,

$$W_t = W\left[1 - \frac{(1+i)^t - 1}{(1+i)^n - 1}\right],$$

or

$$W_t = W\left[\frac{(1+i)^n - (1+i)^t}{(1+i)^n - 1}\right]. \qquad (2)$$

Illustrative Example. What is the wearing value, after 10 years of service, of a locomotive costing new \$10,500, with estimated scrap value \$500 and probable life 25 years, on a 4% basis?

Solution. The wearing value is \$10,000, and the annual depreciation charge will be

$$D = \$10{,}000 \times \left(\frac{1}{s_{\overline{25}|}} \text{ at } 4\%\right)$$
$$= \$10{,}000 \times .02401196$$
$$= \$240.120.$$

By formula (1),

$$W_{10} = \$10{,}000 - \$240.120 \times (s_{\overline{10}|} \text{ at } 4\%)$$
$$= \$10{,}000 - \$240.120 \times 12.006107$$
$$= \$7117.095.$$

If formula (2) had been used, the values of $(1.04)^{25}$ and $(1.04)^{10}$ could have been taken directly from the table of compound amounts, but a troublesome division would have been necessary before the final result could be reached.

If we give to t the values $0, 1, 2, \cdots, n$, in formula (1), we obtain a corresponding series of values for W_t, from which we can construct the *curve of the wearing values*. In the locomotive problem, for example, when the values of the known numbers are substituted, formula (1) becomes

$$W_t = \$10{,}000 - \$240.120 \times s_{\overline{t}|},$$

or

$$W_t = \$10{,}000 - \$240.120\,\frac{(1.04)^t - 1}{.04}.$$

The latter of these two forms shows that the curve is an exponential curve with base 1.04. The computation is most easily managed by using the first form. Giving to t the values 0, 1, 2, 3, · · ·, 25, we find the corresponding values of W_t in round numbers, as in the following table:

t	W_t
0	10,000
1	9,760
2	9,510
3	9,251
4	8,981
5	8,700
6	8,407
7	8,103
8	7,787
9	7,459
10	7,117
11	6,762
12	6,391
13	6,008
14	5,608
15	5,192
16	4,759
17	4,310
18	3,842
19	3,356
20	2,850
21	2,324
22	1,776
23	1,208
24	615
25	0

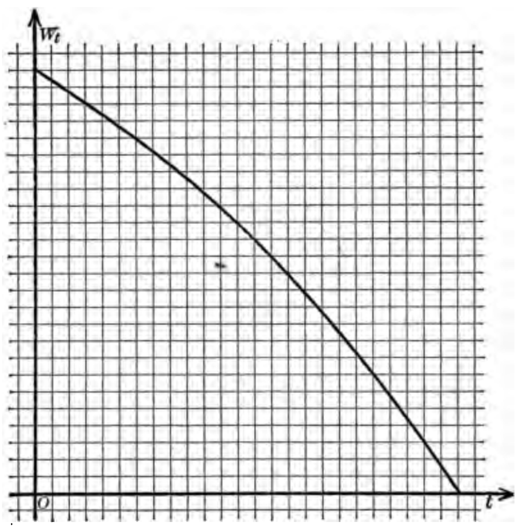

FIG. 8

The curve will then have the form indicated in Fig. 8, where one of the small spaces on the t-axis represents one year, and one space on the vertical axis represents \$400.

EXAMPLES

1. Find the composite life, on a 3% basis, of a plant consisting of three parts, A, B, and C, as follows: A cost new \$10,000, scrap value \$600, probable life 30 years; B cost new \$5000, scrap value \$100, probable life 25 years; C cost new \$8000, scrap value \$300, probable life 20 years.

2. Find the composite life, on a 4% basis, of the plant in Example 1 of § 56.

3. On a 3% basis find the composite life of the plant in Example 4 of § 56. What is the rate of depreciation?

4. The "*condition per cent*" is defined as the ratio of depreciable value existing to the depreciable value new, i.e. $\frac{W_t}{W}$, where W and W_t are the depreciable values new and at the end of t years respectively. Prove that if r denote the condition per cent, theoretically, its value would be given by the formula

$$r = 1 - \frac{s_{\overline{t}|}}{s_{\overline{n}|}} = \frac{(1+i)^n - (1+i)^t}{(1+i)^n - 1}.$$

5. The *composite age* of an asset is the length of time in which the asset would theoretically reach a given per-cent condition, or, what amounts to the same thing, a certain existing wearing value. If t denote the composite age, prove by means of the formula of example (4) that, theoretically, the composite age of an asset having a probable life of n years and having condition per cent r is given by the formula

$$t = \frac{\log\left[(1+i)^n(1-r) + r\right]}{\log(1+i)}.\text{*}$$

6. If W_1, W_2, W_3, be the wearing values of the parts of a plant consisting of three parts, and n_1, n_2, n_3, their probable lives, prove that an approximate value for the composite life n is given by the formula

$$n = \frac{n_3(W_1 + W_2 + W_3)}{W_1 \frac{n_3}{n_1} + W_2 \frac{n_3}{n_2} + W_3},$$

where n_3 is the life of the most durable part.

Suggestion. Formula (4) of § 56 is equivalent to

$$(W_1 + W_2 + W_3)\frac{1}{s_{\overline{n}|}} = W_1 \frac{1}{s_{\overline{n_1}|}} + W_2 \frac{1}{s_{\overline{n_2}|}} + W_3 \frac{1}{s_{\overline{n_3}|}}.$$

Replace $\frac{1}{s_{\overline{n}|}}$, $\frac{1}{s_{\overline{n_1}|}}$, $\frac{1}{s_{\overline{n_2}|}}$, and $\frac{1}{s_{\overline{n_3}|}}$ by their values, expand powers of $1 + i$, and drop powers of i above the first. What is the significance of the formula thus obtained?

7. In § 34 it was shown that

$$C_\infty = C + \frac{C}{i} \cdot \frac{1}{s_{\overline{k}|}}$$

is the capitalized cost of an asset with depreciable value C and probable life of k years. Prove that computing an annual return at rate i on C_∞ would be exactly equivalent to allowing interest on the original investment at rate i and allowing for depreciation by formula (2) of § 56.

* For the formula in Example 5 I am indebted to Mr. Edwin Gruhl, formerly with the Wisconsin Railroad Commission.

Note on depreciation. Any theory of depreciation must of necessity be based largely on conjecture, and any determination of the amount of depreciation is merely an estimate based upon the assumption that present conditions will continue — an assumption which is by no means justified by experience. Indeed, with the exception of Matheson's book on "Depreciation of Factories, Mines, and Industrial Undertakings, and their Valuation," published in 1884, the subject received little attention prior to 1895. Since that time, however, owing partly to a closer study of business methods and partly to the creation of public-utility commissions with power to adjust rates, the subject has held an important place in technical and engineering literature. To one reading the discussions of the subject it would seem that nearly as many theories have been proposed as there have been writers on the subject. The difficulty begins at the very outset, by reason of the fact that writers do not agree upon the items that should be covered by depreciation.

In the foregoing discussion the only item that has been included is the cost of renewal of the asset. It has been assumed that repairs would be provided for in some other way, and that destruction by fire or other extraordinary means would be provided for by insurance. The discussion takes no account of diminished efficiency due to length of service, nor of the value of the output. The only two things that have been taken into consideration are the cost of renewal and the time at which that cost must be incurred. With these two elements supposed to be known, and with all others excluded, the problem of depreciation is perfectly simple and perfectly clear.

The depreciation problem is one of great importance, not only to the manufacturer and to the man engaged in transportation, but to the state as well. It enters into every discussion having to do with the making of rates, whether the rate be the price of transporting passengers or freight of any sort, or whether it be the price of gas or electric current. The various public-utility commissions must make some sort of determination of depreciation, whether an adequate theory exists or not. The practice that has obtained with large manufacturing concerns of "writing off" from their books each year a fixed amount, say $2\frac{1}{2}\%$ for buildings and 5% for machinery, is, after all, only a makeshift.

It is scarcely necessary to warn the novice that *any purely theoretical determination of depreciation may come very wide of the actual facts.*

59. The valuation of mining properties. When a sum of money is loaned, the person making the loan not only receives interest at a stipulated rate at the end of each year, but at the end of a stated period he receives his original capital back again in full. There are, however, some forms of investment where the original

capital is not returned to the investor. In such cases provision must be made for the redemption of the capital by setting aside some portion of the annual income as a redemption fund. A mine is a typical example of this sort of investment, for from it a definite amount of mineral can be removed, and when this is done, the mine must be abandoned.

It is the business of the mining engineer to determine the amount of mineral that can be taken from a mine and the cost of getting the mineral to market. When these elements are determined, together with the rate at which the mineral is to be removed, the problem of determining the *value of the mine* becomes quite definite. The net annual income is the rent of an annuity, and the value of the mine is nothing but the present value of the annuity.

To understand the problem more clearly, let us consider the net annual income of the mine as annual rent of an annuity certain with a term of n years. Let R denote the net annual income, i the rate of interest, and V_n the value of the property. We have, then,

$$V_n = Ra_{\overline{n}|} = R\frac{1-v^n}{i}.$$

If the investor is to receive interest at the rate i, the annual interest will be

$$V_n i = R(1-v^n).$$

The surplus income,

$$R - V_n i = Rv^n,$$

will be available as a payment into the redemption fund. These annual payments, accumulated for n years, will amount to

$$\begin{aligned} Rv^n s_{\overline{n}|} &= Rv^n\frac{(1+i)^n-1}{i} \\ &= R\frac{1-v^n}{i}, \end{aligned} \tag{1}$$

which is exactly the value of the mine as originally determined. We may conclude, then, that the surplus income, accumulated to the end of the term, will exactly provide for the redemption of the original value, *provided the payments into the redemption*

fund can be accumulated at the rate of interest to be received by the investor.

If, however, the redemption fund must be accumulated at a lower rate, as is usually the case, the situation is quite different. The rate of interest received by the investor has been called the *stipulated rate*, while the rate at which the redemption fund can be accumulated is called the *practicable rate.** The determination of the value of the mine resolves itself into the following

PROBLEM. *To find the value of a mine when the rate to be received by the investor differs from the rate at which the redemption fund can be accumulated.*

Let V_n denote the value sought, R the net annual income, i' the stipulated rate, i the practicable rate, and n the time to elapse before the mine is exhausted. The interest paid to the investor annually will be $V_n i'$, and the surplus available for the redemption fund payment will be

$$R - V_n i'.$$

If the redemption-fund payments be accumulated for n years, the amount will be

$$(R - V_n i')\, s_{\overline{n}|},$$

and by hypothesis this amount must be the original value, V_n, of the mine. We have, then,

$$(R - V_n i')\, s_{\overline{n}|} = V_n. \tag{2}$$

Solving equation (2) for V_n, we find

$$V_n = \frac{R}{\dfrac{1}{s_{\overline{n}|}} + i'} = \frac{R}{\dfrac{1}{a_{\overline{n}|}} + i' - i}, \tag{3}$$

where, throughout the discussion, $s_{\overline{n}|}$ is the amount at rate i.

To compute V_n we take the value of $\dfrac{1}{s_{\overline{n}|}}$ directly from the table for the annuity that 1 will purchase, and complete the computation by means of logarithms. Extensive tables, giving the value of V_n directly for practically every case that would occur, have been published by Hoskold in the work cited above.

*Hoskold, The Engineer's Valuing Assistant. London, 1905.

EXAMPLES

1. A coal mine can be made to yield \$10,000 net annually for 25 years. What is its value if the stipulated rate is 7% and the redemption fund can be accumulated at 3%?

Solution. From the table we find

$$\frac{1}{s_{\overline{25|}}} = .0274279.$$

Then

$$V_{25} = \frac{10{,}000}{.0274279 + .07}$$

$$= \frac{10{,}000}{.0974279}$$

$$\log V_{25} = 5.0113167$$

$$V_{25} = \$102{,}640.$$

2. Verify the result in Example 1 by showing that the surplus income, after 7% has been deducted for the investor, will accumulate at 3% to the value V_{25}, as found.

3. It is estimated that from a certain mine the mineral will be exhausted in 20 years, and that during that time it can be made to yield \$5000 per year net. What should be the purchase price to yield 8% to the investor if the redemption fund can be accumulated at $3\frac{1}{2}\%$.

CHAPTER X

BUILDING AND LOAN ASSOCIATIONS

60. Definitions and first principles. A *building and loan association* is an association of persons formed for the purpose of enabling its members to accumulate money by periodical payments into its treasury, to be invested from time to time in loans to those of its members who wish to build homes. The membership of such associations usually consists of two classes: shareholders who are investors only, and shareholders who are at the same time investors and borrowers.

There are many plans for building and loan associations, but most of them require a small monthly or weekly payment, which in the case of borrowers provides for the payment of interest and for the establishment of a sinking fund which will extinguish the indebtedness when the stock *matures*, and in the case of investors constitutes a savings account which is augmented from time to time by the dividends which may be declared out of profits earned by the association. The sources of profits are interest earned on loans made to borrowing members; gains due to the difference between the *book value* and the *withdrawal value* of stock belonging to members who retire before their stock has matured; fines assessed upon members who may have been delinquent in the payment either of dues or of interest; fees charged on new business; and, finally, any undivided surplus that may have remained over at the last distribution.*

The *net profits* are the profits remaining after all the expenses of conducting the business have been paid. The net profits are

* One other source of profit frequently utilized by loan associations is the difference between the interest received on money borrowed by the association at a lower rate than that charged to borrowing shareholders, and the interest paid on loans thus made. The amount of such loans is usually limited by law. For example, in Wisconsin the limit is 20% of the assets of the association.

sometimes distributed on the basis of the book values immediately after the last distribution of profits, and sometimes the basis of distribution is found by adding to the book values, immediately after the last distribution, a portion of the amount paid in dues since the last distribution. The latter plan is more equitable, since each shareholder's profits are more nearly proportional to the amount of money he has invested. The former plan will give a slightly higher rate of profit.

PROBLEM. *To determine the rate of profit for a period.*

The rate of profit for a period cannot be determined in advance, since it depends upon elements that cannot be known until the end of the period. For example, loans in force at the beginning of the period may be repaid within the period, or a considerable amount of stock may be withdrawn, or part of the funds of the association may remain uninvested. At the end of the period when the elements of the problem are all known, it reduces to a simple problem in percentage.

Let I denote the total interest received during the period, f the amount received from fees and fines, and e the total expenses and losses. Further, let B be the book value immediately after the last distribution of all stock remaining in force to the end of the period, and d the amount added to the book values to enable the amount paid in as dues during the period to participate equitably in the profits. The net profits N will then be

$$N = I + f - e, \tag{1}$$

and the rate of profit r will be

$$r = \frac{N}{B} = \frac{I+f-e}{B} \tag{2}$$

if the distribution is made on the basis of book values at the beginning of the period, or

$$r = \frac{N}{B+d} = \frac{I+f-e}{B+d} \tag{3}$$

if dues paid in during the period are allowed to participate in the profits.

To find d, one has, in effect, a problem in the equation of payments for every shareholder. If, for example, a shareholder pays \$1 a month per share, and pays promptly on the first of the month for six months, each share of his stock will contribute \$3.50 toward d. The stock of a shareholder delinquent in all dues for the period would contribute nothing to d. The stock of a member who paid \$1 the first month, \$1 the second month, nothing the third and fourth months, \$3 the fifth month, and \$1 the sixth month, would contribute \$3 to d, since, by the approximate rule for equation of accounts, \$1 for six months, \$1 for five months, \$3 for two months, and \$1 for one month would be equivalent to \$3 for six months.

It would be impossible to give even a concrete example of the determination of the rate of profit without taking into account the receipts and disbursements, and the status of every shareholder's stock for an entire period.

The rate of profit once determined, the apportionment of the net profits to each shareholder's stock is a simple matter.

EXAMPLES

1. B holds 10 shares in the Provident Building and Loan Association, which, immediately after the distribution of profits on January 1, 1910, had a book value of \$72.98 per share. During the following half-year period he paid as follows: January 1, \$10; February 1, \$10; May 1, \$30; June 1, \$10. On July 1 the rate of profits was .0254. What was his share of the profits?

Solution. To find the amount of the dues participating in the profits, we note that B had invested \$10 for 6 months, \$10 for 5 months, \$30 for 2 months, and \$10 for one month. These sums would be equivalent to a sum x for 6 months, where x is determined from the equation

$$\begin{aligned} 6x &= 10 \times 6 + 10 \times 5 + 30 \times 2 + 10 \times 1 \\ &= 180. \end{aligned}$$

Therefore $x = 30.$

The profits on B's stock will then be computed on \$729.80 + \$30, or \$759.80. Consequently, B's share of the profits will be

$$\$729.80 \times .0254 = \$18.54.$$

2. A has 10 shares in a building association, and on January 1, 1911, these shares were worth \$63.92 per share. During the next six months he paid dues as follows: January 15, \$10; February 15, \$10; May 15, \$30; June 15, \$10. What was his share of the profits for the period from January 1 to July 1?

61. The effective rate of interest realized by the investor.

PROBLEM. *To find the effective rate of interest realized by the investor when the maturing value of the stock, the monthly payment, and the time of maturity are supposed to be known.*

The problem is similar to that of § 41, where the rate of interest paid by an annuity was determined. Let C be the value at maturity, M the monthly payment, and n the time from the date of the first payment to maturity. The first payment is on interest for n years, and the remaining payments constitute an annuity of $12M$ per annum for n years. Consequently,

$$C = M(1+i)^n + M\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{12}} - 1}. \tag{1}$$

The problem is, then, to determine i from equation (1). After easy reductions equation (1) takes the form

$$\frac{C}{M} = \frac{(1+i)^{n+\frac{1}{12}} - 1}{(1+i)^{\frac{1}{12}} - 1}. \tag{2}$$

To illustrate the method used in finding an approximate value of i, suppose that $C = \$200$, $M = \$1$, $n = 12$ years. Equation (2) then becomes

$$200 = \frac{(1+i)^{12+\frac{1}{12}} - 1}{(1+i)^{\frac{1}{12}} - 1}. \tag{3}$$

To obtain a first approximation, we note that the rate is almost certainly less than the rate paid by the borrowers, since expenses and losses have to be met out of interest received from borrowers. At present, loan associations are charging about 6% on loans. A first guess would be something less than 6%. Moreover, at 6%, when $M = 1$, the first term of the right member of (1) would be

\$2.012, and if the annuity were payable annually, the second member would be \$202.44, so that we are reasonably sure that i lies between .05 and .06.

Let $$i = .055 + h.$$

We know that h is a small number, certainly less than .01, so that the binomial expansions for $(1.055 + h)^{12 + \frac{1}{12}}$ and $(1.055 + h)^{\frac{1}{12}}$ will converge rapidly. Equation (3) takes the form

$$200 = \frac{(1.055 + h)^{12 + \frac{1}{12}} - 1}{(1.055 + h)^{\frac{1}{12}} - 1}. \tag{4}$$

Expanding the powers of the binomials in (4) and dropping powers of h above the first, we find

$$\begin{aligned} 200 &= \frac{(1.055)^{\frac{145}{12}} + \frac{145}{12}(1.055)^{\frac{133}{12}} h - 1}{(1.055)^{\frac{1}{12}} + \frac{1}{12}(1.055)^{-\frac{11}{12}} h - 1} \\ &= \frac{1.90971 + 21.8726\, h - 1}{1.00447 + .07934\, h - 1}. \end{aligned}$$

Clearing of fractions,

$$.89400 + 15.868\, h = .90971 + 21.8726\, h.$$

Solving for h, $$\begin{aligned} h &= -\frac{.0157}{6.0046} \\ &= -.0026+. \end{aligned}$$

This value of h gives $$i = .0524 \ldots.$$

Starting with $i = .0524$, we find the second approximation in exactly the same way.

EXAMPLE

A man pays \$2 to a building and loan association on the first of each month for 7 years, when he receives \$200 as the face value of his stock. What rate of interest does he receive?

62. The time required for stock to mature.

PROBLEM. *To find the approximate time in which stock will mature when the maturing value, the monthly payment, and the approximate rate of interest are known.*

Let C be the maturing value, M the monthly payment, i the approximate rate of interest, and n the unknown time. Then, as in § 61, we have the equation

$$C = M(1+i)^n + M\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{12}} - 1}, \tag{1}$$

from which to determine the time n. Transforming (1) by dividing by M and reducing the right member to a common denominator, we have

$$\frac{C}{M} = \frac{(1+i)^{n+\frac{1}{12}} - 1}{(1+i)^{\frac{1}{12}} - 1}. \tag{2}$$

Solving (2) for the expression containing n, we find

$$(1+i)^{n+\frac{1}{12}} = 1 + \frac{C}{M}[(1+i)^{\frac{1}{12}} - 1].$$

Taking logarithms of both sides and dividing by $\log 1+i$,

$$n = -\frac{1}{12} + \frac{\log\left\{1 + \frac{C}{M}[(1+i)^{\frac{1}{12}} - 1]\right\}}{\log(1+i)}. \tag{3}$$

If we remember that the expression $12[(1+i)^{\frac{1}{12}} - 1]$ has been defined as $j_{(12)}$ (see (3′), § 23), equation (3) takes the form

$$n = -\frac{1}{12} + \frac{\log\left(1 + \frac{C}{12M} j_{(12)}\right)}{\log(1+i)}. \tag{4}$$

If the nominal rate, payable twice a year, as is customary, is given, we have only to replace $1+i$ by $\left(1+\frac{j}{2}\right)^2$, and formula (3) becomes

$$n = -\frac{1}{12} + \frac{\log\left\{1 + \frac{C}{M}\left[\left(1+\frac{j}{2}\right)^{\frac{1}{6}} - 1\right]\right\}}{2\log\left(1+\frac{j}{2}\right)}, \tag{5}$$

or

$$n = -\frac{1}{12} + \frac{\log\left[1 + \frac{C}{6M}\cdot\left(\frac{j}{2}\right)_6\right]}{2\log\left(1+\frac{j}{2}\right)}. \tag{5'}$$

EXAMPLES

1. A building and loan association issues stock maturing to $200 per share, with a monthly payment of $1 per share. Four successive semi-annual distributions of profits were at rates .0287, .0283, .0254, and .0307. About how long will it take the stock to mature?

Solution. The arithmetical mean of the rates for the two years is .0283, so that we may put $j = .0566$. Moreover, $C = 200$ and $M = 1$, so that formula (5) becomes

$$n = -\frac{1}{12} + \frac{\log\{1 + 200\,[(1.0283)^{\frac{1}{6}} - 1]\}}{2\log(1.0283)}.$$

Using logarithms, we find

$$\log(1.0283) = .0121198$$

$$(1.0283)^{\frac{1}{6}} = 1.00466.$$

Consequently,

$$\begin{aligned} n &= -\frac{1}{12} + \frac{\log 1.9324}{2\log(1.0283)} \\ &= -\frac{1}{12} + \frac{.2860970}{.0242396} \\ &= -\frac{1}{12} + 11.80 \\ &= 11.72+. \end{aligned}$$

The required time is therefore 11 years, 8 months, and 19 days.

2. Suppose the association in Example 1 decides to issue stock on which payments of $2 per month per share are made. How long will it take for such stock to mature to $2 per share?

63. The rate of interest paid by the borrowing shareholder. We may look upon the repayment of a loan made to a borrowing shareholder in two ways:

In the first place, we may consider the loan as an entirely distinct transaction which has nothing whatever to do with the ownership of stock in the association. From this point of view the rate of interest is simply the effective rate corresponding to the nominal rate charged by the association and payable twelve times a year.

To find the rate when the loan is considered as a separate transaction, we may assume that the interest is payable at the end of each month instead of at the beginning. The effective rate is, then,

$$i = \left(1 + \frac{j}{12}\right)^{12} - 1.* \tag{1}$$

* If interest is payable strictly in advance, the problem of determining the effective rate is interesting from the theoretical standpoint, since it leads to the consideration of an infinite geometrical progression. To find the effective rate under such

On the other hand, we may consider the transaction as the repayment of a loan, principal and interest, in equal monthly installments. For example, a shareholder borrows $4000 from an association which issues stock in shares of $200 each, matured by the payment of $1 per month per share, and which charges interest at 6% nominal, payable monthly in advance. He must take out 20 shares of stock, for which he pays $20 a month in dues. Besides, he must pay $240 a year in twelve monthly installments of $20 each for interest, making $40 a month in all. The $40 per month may be considered as a series of payments that will extinguish the debt when the stock matures. Looked at from this point of view the determination of the rate of interest is a more difficult matter.

PROBLEM. *To determine the rate of interest paid by a borrowing shareholder when the monthly interest payment and the monthly dues are together considered as a single sum for the repayment of the principal and interest by a series of monthly installments.*

The problem differs from that of § 42 only in that in the present case the annuity is "due"; i.e. the first payment is made at the beginning of the first period instead of at the end. The problem can be considered from the standpoint of a single share of stock, since if it requires the value at maturity of m shares of stock, m will be a common factor which may be divided out from both sides of the equations involved.

circumstances, we must note that at the end of the first period the investor would have the following sums: (1) The original unit of principal; (2) the interest on the principal at rate $\frac{j}{p}$ per period; (3) the interest on the interest $\frac{j}{p}$ at rate $\frac{j}{p}$ per period; (4) the interest on the interest, on the interest, and so on indefinitely.

These amounts form an infinite geometrical progression, $1+\frac{j}{p}+\frac{j^2}{p^2}+\frac{j^3}{p^3}+\cdots$, whose sum is $\left(1-\frac{j}{p}\right)^{-1}$. At the end of the second interval the amount on each dollar of principal available at the beginning of the period would be again $\left(1-\frac{j}{p}\right)^{-1}$, and the amount of $\left(1-\frac{j}{p}\right)^{-1}$ would be $\left(1-\frac{j}{p}\right)^{-2}$. Proceeding in this way, we should find for the amount at the end of the year, i.e. at the end of the pth period, $\left(1-\frac{j}{p}\right)^{-p}$, and the effective rate would be $i=\left(1-\frac{j}{p}\right)^{-p}-1$.

Let C be the value to which a share matures, M the monthly due, n the number of years required for stock to mature, k the nominal rate charged on loans, and i the unknown rate of interest paid by the borrower.

The monthly interest payment will be $\frac{Ck}{12}$, so that the total monthly payment will be

$$\frac{Ck}{12} + M.$$

The present value of the annuity will be

$$C = \frac{Ck}{12} + M + 12\left(\frac{Ck}{12} + M\right) a_{\overline{n}|}^{(12)}. \tag{2}$$

If, for the sake of brevity, we write

$$\frac{Ck}{12} + M = M_1, \tag{3}$$

M_1 will be the total monthly payment, and equation (2) reduces to the form

$$\frac{C - M_1}{12\,M_1} = \frac{1 - (1+i)^{-n}}{12[(1+i)^{\frac{1}{12}} - 1]}, \tag{4}$$

where $a_{\overline{n}|}^{(12)}$ has been replaced by its value from equation (3) of § 32.

Suppose we know that the value of i is approximately equal to a known value, i'; then

$$i = i' + h, \tag{5}$$

where h is numerically small as compared with i', which is itself less than .1 under actual conditions. By virtue of (5), equation (4) takes the form

$$\frac{C - M_1}{12\,M_1} = \frac{1 - (1 + i' + h)^{-n}}{12[(1 + i' + h)^{\frac{1}{12}} - 1]}. \tag{6}$$

If the powers of $1 + i' + h$ occurring in (6) be expanded in powers of h by the binomial theorem, and all powers of h higher than the first be dropped, we have for the *approximate* determination of h the equation

$$\frac{C - M_1}{12\,M_1} = \frac{1 - (1+i')^{-n} + n(1+i')^{-(n+1)}h}{12[(1+i')^{\frac{1}{12}} - 1] + (1+i')^{-\frac{11}{12}}h}. \tag{7}$$

If h' be the value of h determined from equation (7), we shall have for the corresponding value i'' of i

$$i'' = i' + h'. \tag{8}$$

Using i'' in place of i' in equation (6), the process may be repeated if a closer approximation is desired.

To illustrate the process, suppose a man borrows of an association which charges 6% nominal, payable monthly in advance, and whose shares mature to \$200 in $11\frac{1}{2}$ years by a monthly payment of \$1 per share. The amount borrowed on one share would be \$200, and on this amount the interest at 6% would be \$12 a year, or \$1 per month. The total amount paid each month is, then, \$2; i.e. $M_1 = 2$. Consequently, equation (4) becomes

$$\frac{33}{4} = 8.25 = \frac{1-(1+i)^{-\frac{23}{2}}}{12\left[(1+i)^{\frac{1}{12}}-1\right]}. \tag{9}$$

The problem is now to determine an approximate value of i.

To obtain a first approximation for i from the tables, we note that the present value of the annuity of 1 per annum for $11\frac{1}{2}$ years at 6% would be approximately the arithmetical mean of the present values for 11 and for 12 years, or \$8.14. Similarly, the present value of an annuity of 1 per annum, payable annually for $11\frac{1}{2}$ years at 7%, would be approximately \$7.72. Moreover, as the frequency of payment increases, the present value of the annuity will be slightly increased. From these considerations we guess that i lies between .06 and .07.

Assuming that $i' = .06$, we have

$$i = .06 + h,$$

where h is a small positive number. Substituting $.06 + h$ for i in (9), expanding, and dropping powers of h above the first, we find for the approximate determination of h the equation

$$\begin{aligned} 8.25 &= \frac{1-(1.06)^{-\frac{23}{2}}+\frac{23}{2}(1.06)^{-\frac{25}{2}}h}{12\left[(1.06)^{\frac{1}{12}}+\frac{1}{12}(1.06)^{-\frac{11}{12}}h-1\right]} \\ &= \frac{.48834+5.55104\,h}{.05844+.94799\,h}. \end{aligned}$$

The last equation, when cleared of fractions, becomes

$$.48213 + 7.82092\,h = .48834 + 5.55104\,h;$$

whence $$2.26988\,h = .00621.$$

From this equation $$h = .00007.$$

The approximate value of i is therefore

$$i = .06 + .0027$$
$$= .0627.$$

ILLUSTRATIVE EXAMPLE. A man borrows \$200 from a loan association charging 6% nominal, payable monthly in advance, and finds that his one share matures to \$200 in exactly 12 years. What rate of interest did he pay if the monthly payment of \$2, consisting of due and interest, be considered as one of the equal monthly installments for repaying the debt, principal and interest?

EXAMPLES

1. A building and loan association has \$142,951.74 loaned to its members. The book value of its stock is \$119,953.19. The expenses for the period from June 30, 1909, to December 31, 1909, were \$607.69. Fees, fines, and other profits amounted to \$108.32. The association charges 6% on its loans. Assuming that all interest due was paid, and that no stock was withdrawn during the period, what was the rate of profit for the half year?

2. A man paid \$5 per month to a building and loan association for 11 years and 9 months, when his stock matured to \$1000. What rate of interest did he receive?

3. An association is able to pay a semiannual dividend of .0275 on the dollar. How long will it take a share of stock on which \$1 per month is paid to mature to \$200?

4. If a man is able to pay off a mortgage of \$2000 by paying \$20 to a building and loan association in 145 monthly payments, what effective rate of interest is he paying on his loan?

5. Two men each borrow \$2000 from a building and loan association which charges 6% nominal, payable monthly in advance. One man pays \$20 and the other \$30 per month. If the association pays a semiannual dividend of $2\frac{1}{2}$% on book values, how long will it take for each man to pay his loan, and which pays the greater rate of interest when the transactions are looked upon from the standpoint of the payment of the loans, principal and interest, by equal monthly installments?

PART III. PROBABILITY AND ITS APPLICATIONS TO FINANCIAL PROBLEMS

CHAPTER XI

THE THEORY OF PROBABILITY

64. Definitions and first principles. If an ordinary die, with faces numbered from one to six points, be thrown, there are six possible positions in which it may come to rest; i.e. it may fall with any one of its six faces uppermost. Moreover, if the die is a perfect cube made of homogeneous material, it is equally likely that any one of its six faces will fall uppermost. There are six *cases*, or six *events*, in question. If we are interested in having the die fall with a given face uppermost, we say that one case is favorable to us and five unfavorable. In everyday language, we say that there is one chance in six in our favor.

Again, if a bag contains four white balls and two black ones, and we wish to draw out a white ball, there are six possible cases, of which four are favorable and two unfavorable.

We may say that it is probable that in a single trial a white ball would be drawn, but it would be impossible to draw conclusions without a carefully framed definition to start with. The following definition is made the basis of the theory of probability.

The probability that, among several equally likely events, a given event will happen is the ratio of the number of favorable cases to the whole number of possible cases.

Thus, in the first example, the probability that the die will fall with one point uppermost is $\frac{1}{6}$, since one case among six possible ones is favorable. In the second example the probability that in a single trial a white ball will be drawn is $\frac{4}{6}$, or $\frac{2}{3}$.

The foregoing definition is purely arbitrary and must be taken to mean that if the same situation were to come into existence a very great number of times, the ratio of the number of times it will actually happen to the number of times the situation recurs will be *very near* the ratio of the number of the favorable cases to the whole number of possible cases. For example, if a single die be thrown 6000 times, we expect it to fall with one point uppermost 1000 times. If, however, the trial were to be made, the result might be different. The probability that a coin will fall with "heads" uppermost is $\frac{1}{2}$, but in an actual trial in which 100 throws were made, the coin fell with heads uppermost 42 times, and not 50 times, as we should have expected. Nevertheless, we believe that *in the long run* the frequency with which a given face of the die would fall uppermost would be one sixth of the total number of throws, and the frequency with which the coin would fall "heads" would be one half the total number of trials.

The restriction that all the cases must be equally probable is a fundamental one, as a single example will show. We might say with absolute certainty that a given child ten years of age will either die or not die within a year. There are, then, two events in question — living one year and dying within the year; and we might say that the probability that the child will live one year is $\frac{1}{2}$. This would be very far from the truth, however, for from a very large number of records it has been found that out of 100,000 children who under certain conditions have reached the age of ten years, only 749 die within the year. The probability that a healthy child ten years of age will die before reaching his eleventh birthday is .00749, and not .5. The error consisted in assuming that living one year and dying within the year are equally likely for a child aged ten years.

If the number of favorable cases is a and the number of unfavorable cases is b, the total number of cases is $a + b$, and the probability that an event will happen takes the mathematical form

$$p = \frac{a}{a+b}. \qquad (1)$$

If there are no unfavorable cases, $b = 0$ and

$$p = \frac{a}{a+0} = 1.$$

Moreover, the event is certain to happen. *Certainty* is therefore expressed by 1.

If there are no favorable cases, the event is impossible, so that the expression for *impossibility* is

$$p = \frac{0}{0+b} = 0.$$

Two events are said to be *complementary* if the happening of one excludes the possibility of the other, and the sum of their probabilities is 1. For example, if we draw a ball from a bag containing four white and two black balls, the probability of drawing a white ball is $\frac{2}{3}$ and the probability of drawing a black ball is $\frac{1}{3}$. Moreover, the two events cannot happen at the same time. They are therefore complementary.

THEOREM. *If the probability that an event will happen is p, the probability that it will fail is $1-p$.*

Proof. Let q be the probability of failure. Then

$$p = \frac{a}{a+b} \quad \text{and} \quad q = \frac{b}{a+b}.$$

The happening and the failure of the event are then complementary, since they are mutually exclusive, and

$$p + q = \frac{a}{a+b} + \frac{b}{a+b} = 1;$$

consequently,

$$q = 1 - p, \tag{2}$$

as was to be proved.

65. Simple problems in probability. Many of the simpler problems in the theory of probability may be solved by means of the definition of probability, and the fundamental theorems and formulas from the theory of permutations and combinations. The truth of the following propositions concerning permutations and combinations will be assumed.

1. *If one act can be performed in p ways, and if, after this act is completed, a second unrelated act can be performed in q ways, the number of ways in which the two acts can be performed in succession is pq.*

2. *The number of permutations of n things taken r at a time is*

$$A\binom{n}{r} = \frac{\underline{|n}}{\underline{|n-r}}, \tag{1}$$

and for the special case where $n = r$

$$A_n = \underline{|n}. \tag{2}$$

3. *The number of combinations of n things taken r at a time is*

$$C\binom{n}{r} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. \tag{3}$$

EXAMPLES

1. A die is thrown once. What is the probability that the number of points is less than 5?

Suggestion. There are four favorable cases.

2. Ten balls, exactly alike except that they are numbered from 1 to 10, are put into a bag, and a single ball is drawn at random. What is the probability that the ball is numbered 1? What is the probability that the ball is numbered either 1 or 2?

3. If two of the balls described in Example 2 are drawn simultaneously, what is the probability of drawing the pair numbered 3 and 5?

Suggestion. There are $C\binom{10}{2} = 45$ ways of selecting a pair from 10 numbered balls.

4. A bag contains n balls numbered consecutively from 1 to n, and from it three are drawn simultaneously at random. What is the probability that the numbers are 1, 2, and 3?

5. Two coins are tossed into the air at the same time (or in succession). What is the probability that both will fall heads?

Suggestion. By proposition 1, two coins may fall in any one of four ways.

6. What is the probability that 10 coins tossed into the air at the same time will all fall heads? What is the probability that a single coin tossed into the air 10 times will fall heads every time?

7. Two balls are drawn at the same time from a bag containing 3 white and 5 black balls. What is the probability that both will be white? that both will be black? that one will be black and one white?

Suggestion. There are $C\binom{8}{2}$ possible pairs of balls and $C\binom{3}{2}$ possible pairs of white balls.

8. What is the chance of throwing one, and only one, 5 with one throw of two dice?

66. Total and partial probability. The probability that *a given event* in a series of n mutually exclusive events will happen and all the others fail is called *partial*, or *relative*, *probability*, and the probability that *any event whatever* of the series will happen and the others fail is called *total probability.*

To illustrate, suppose a bag contains 11 white balls, of which 5 are marked with crosses and 6 not; 7 black balls, 4 with and 3 without crosses; 7 yellow balls, 2 with and 5 without crosses. The drawing of a ball with a cross occurs with any one of a series of three independent events: viz., (1) drawing a white ball with a cross, (2) drawing a black ball with a cross, (3) drawing a yellow ball with a cross. The probability that a given one of these three events — for example, drawing a white ball with a cross — will happen is partial probability, while the probability of drawing a ball with a cross regardless of color is total probability.

In the foregoing examples there are four things that interest us:

1. The probability of drawing a white ball with a cross is $\frac{5}{25}$.
2. The probability of drawing a black ball with a cross is $\frac{4}{25}$.
3. The probability of drawing a yellow ball with a cross is $\frac{2}{25}$.
4. The probability of drawing a ball with a cross is $\frac{11}{25}$.

The striking fact is that the sum of the partial probabilities, $\frac{5}{25}$, $\frac{4}{25}$, and $\frac{2}{25}$, is equal to the total probability. This fact is true in the general case and forms one of the two fundamental rules for the computation of probabilities.

THEOREM. *The total probability of an event is equal to the sum of its partial probabilities.*

Proof. Suppose that there are m possible cases and n events in the series. Let $a_1, a_2, \cdots, a_n$, be the number of cases favorable

to the happening of the respective events of the series, and let $p_1, p_2, \cdots, p_n$, be the probabilities of these events. By definition,

$$p_1 = \frac{a_1}{m}, \quad p_2 = \frac{a_2}{m}, \quad \cdots, \quad p_n = \frac{a_n}{m}. \tag{1}$$

But the total probability that an event of the series will happen is

$$p = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{m} = \frac{a_1}{m} + \frac{a_2}{m} + \cdots + \frac{a_n}{m}, \tag{2}$$

since the number of favorable cases is $a_1 + a_2 + \cdots + a_n$. Replacing the fractions $\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_n}{m}$, in (2) by their values, we have the final result,

$$p = p_1 + p_2 + \cdots + p_n. \tag{3}$$

It is of the utmost importance that the events in question be mutually exclusive. To illustrate, suppose that the probability that A can solve a given problem is $\frac{1}{4}$, and the probability that B can solve it is $\frac{1}{3}$. The probability that the problem will be solved if both work at it is not $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, as we might infer, because the two events are not mutually exclusive. The mutually exclusive events of the series are (1) A succeeds and B fails; (2) A fails and B succeeds; (3) both succeed. It will be subsequently shown that the probabilities of these mutually exclusive events are $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{12}$, so that the total probability that the problem will be solved is

$$\tfrac{1}{6} + \tfrac{1}{4} + \tfrac{1}{12} = \tfrac{1}{2}.$$

On the other hand, if A and B, with others, are competitors in a race, and the probability that A will win is $\frac{1}{4}$, while the probability that B will win is $\frac{1}{3}$, the probability that A or B will win is

$$\tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12},$$

since the two events, A wins and B wins, are mutually exclusive.

67. Compound probability. Two events are said to be *independent* if the happening of one has no influence upon the other, and vice versa. *Compound probability* is the probability that two

independent events will happen simultaneously or in succession. For example, the probability that A and B, working independently, will solve a given problem is compound and, as we shall soon see, depends upon the probabilities that each will solve it. The statement of this relation is the second fundamental proposition in the theory of probability.

THEOREM. *The compound probability of two independent events is the product of the probabilities of the two events taken singly.*

Proof. Let p_1 be the probability of the first event, and p_2 the probability of the second. Suppose that the number of cases possible in connection with the first event is m_1, and the number in connection with the second is m_2. Suppose also that of the m_1 cases a_1 are favorable to the happening of the first event, and of the m_2 cases a_2 are favorable to the happening of the second event. By definition,

$$p_1 = \frac{a_1}{m_1} \text{ and } p_2 = \frac{a_2}{m_2}.$$

Moreover, we may combine every one of the possible cases for the first event with every one for the second, so that, by the first proposition on combinations and permutations (§ 65), the number of possible cases in connection with the simultaneous happening of the two events is $m_1 m_2$. Similarly, we may show that the number of cases favorable to the happening of the two events simultaneously is $a_1 a_2$. If, therefore, p represent the compound probability that the two events will happen simultaneously,

$$p = \frac{a_1 a_2}{m_1 m_2}.$$

But $$\frac{a_1 a_2}{m_1 m_2} = \frac{a_1}{m_1} \cdot \frac{a_2}{m_2} = p_1 \cdot p_2.$$

Therefore $$p = p_1 \cdot p_2. \tag{1}$$

COROLLARY I. If p be the compound probability that n independent events whose separate probabilities are $p_1, p_2, \cdots, p_n$, will happen simultaneously (or in succession),

$$p = p_1 p_2 \cdots p_n. \tag{2}$$

COROLLARY II. If $p_1, p_2, \cdots, p_n$, are the separate probabilities of n independent events, the probability that they will all fail is

$$(1-p_1)(1-p_2)\cdots(1-p_n), \tag{3}$$

and the probability that the first r will happen and the remainder fail is

$$p_1 p_2 \cdots p_r (1-p_{r+1}) \cdots (1-p_n). \tag{4}$$

EXAMPLES

1. The probability that A working alone can solve a given problem is $\frac{1}{4}$, and the probability that B working alone can solve it is $\frac{1}{3}$. What is the probability that the problem will be solved if both work at it, each alone.

Solution. The mutually exclusive events, any one of which would bring about the solution of the problem, are (1) A succeeds, B fails; (2) A fails, B succeeds; (3) both succeed. Moreover, the success or failure of one has nothing to do with the success or failure of the other. The compound probability of the simultaneous success of A and failure of B is, then,

$$\tfrac{1}{4}(1 - \tfrac{1}{3}) = \tfrac{1}{6}.$$

Likewise, the compound probability that A will fail and B succeed is

$$(1 - \tfrac{1}{4})\tfrac{1}{3} = \tfrac{1}{4};$$

and, finally, the probability that both will succeed is

$$\tfrac{1}{4} \cdot \tfrac{1}{3} = \tfrac{1}{12}.$$

The *total* probability that the problem will be solved is the sum of the partial probabilities, or

$$\tfrac{1}{6} + \tfrac{1}{4} + \tfrac{1}{12} = \tfrac{1}{2}.$$

2. A, B, and C run in a race with other competitors. The probability that A will win is $\frac{1}{6}$, that B will win is $\frac{1}{6}$, and that C will win is $\frac{1}{4}$. What is the probability that one of the three will win?

3. Two dice are thrown simultaneously. What is the probability that the throw will be greater than 9?

4. A single die is thrown twice. What is the probability that the first throw will be less than 3 and the second less than 5?

5. The letters of the word "probability" are placed at random in a straight line. What is the probability that two vowels will come together?

68. Probability of an event when several trials are made.

THEOREM I. *The probability that an event will happen exactly r times in n trials is*

$$\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}p^r q^{n-r},$$

where p is the probability that it will happen and q the probability that it will fail in a single trial.

Proof. By (4) of § 67 the *compound* probability that the event will happen in a *given* trial and fail in the other $n-1$ is

$$pq^{n-1}.$$

The *total* probability that it will happen in *some one* of the n trials is the sum of the probabilities that it will happen in the separate trials, viz.

$$pq^{n-1}+pq^{n-1}+\cdots \text{ to } n \text{ terms} = npq^{n-1}.$$

Again, the *compound* probability that an event will happen in two assigned trials and fail in the other $n-2$, say the fifth and the eleventh, is

$$p^2q^{n-2},$$

and the *total* probability that it will happen in any two trials whatever is

$$p^2q^{n-2}+p^2q^{n-2}+\cdots,$$

where the number of terms is equal to the number of ways of specifying two trials out of n, i.e. $C(^n_2)$. But, by proposition (3) of § 65,

$$C(^n_2)=\frac{n(n-1)}{1\cdot 2}.$$

Therefore the number of terms is $\frac{n(n-1)}{1\cdot 2}$, and consequently the total probability that an event will happen twice and fail $n-2$ times in n trials is

$$\frac{n(n-1)}{1\cdot 2}p^2q^{n-2}.$$

In general, the probability that the event will happen in r assigned trials and fail in the other $n-r$ trials is

$$p^r q^{n-r}.$$

But there are

$$C\binom{n}{r}=\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}$$

ways of specifying r trials out of n trials. The total probability that the event will happen in exactly r trials is

$$p^r q^{n-r}+p^r q^{n-r}+\cdots, \text{ to } C\binom{n}{r} \text{ terms.}$$

It is therefore

$$C\binom{n}{r}p^r q^{n-r}=\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}p^r q^{n-r}.$$

THEOREM II. *The probability that an event will happen at least r times in n trials is*

$$p^n+np^{n-1}q+\frac{n(n-1)}{1\cdot 2}p^{n-2}q^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}p^{n-r}q^r,$$

where p is the probability that the event will happen, and q the probability that it will fail in a single trial.

The event will happen r times if it happens n times, or if it happens $n-1$ times, or if it happens $n-2$ times, and so on to $n-(n-r)$ times. Consequently, the required probability is the *total* probability made up of the partial probabilities that it will happen n times, $n-1$ times, and so on. But, by Theorem I, the partial probability that the event will happen n times in n trials is p^n, that it will happen $n-1$ times is $np^{n-1}q$; that it will happen $n-2$ times is $\frac{n(n-1)}{1\cdot 2}p^{n-2}q^2$, and, finally, the probability that it will happen r times is $\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}p^{n-r}q^r$. Consequently, the total probability that it will happen at least r times is

$$p^n+np^{n-1}q+\frac{n(n-1)}{1\cdot 2}p^{n-2}q^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}p^{n-r}q^r.$$

EXAMPLES

1. Find the probability of throwing 1 and only 1 point in two trials with 1 die.

2. Find the probability of throwing at least 1 point in 2 throws of 1 die.

3. If, on an average, 99 out of 100 ships reach port safely, find the probability that at least 2 out of 10 will arrive.

69. Mathematical expectation. The name *mathematical expectation* has been given to the product of a sum whose payment depends upon the happening of some contingent event, multiplied by the probability that the event will happen. For example, if a stake of \$6000 were offered, to be paid if the throw of a die is 1, the mathematical expectation of a player is \$1000. This does not mean, of course, that a man having \$1000 could afford to risk it on the throw of a single die, but rather that, by paying \$1000 for each throw, in a very large number of throws the player would come out approximately even in the long run, in a perfectly fair game.

A second example may serve to make the matter clearer. Suppose that 100,000 men, all aged forty years, agree to make up a fund of such an amount that at the end of a year each survivor may receive one dollar. What is the mathematical expectation of each participant?

It has been found by the American life-insurance companies that, of every 100,000 men aged forty years, approximately 979 will die within a year. The probability that any one of these men will live one year is therefore $\frac{99{,}021}{100{,}000} = .99021$. If we neglect interest, the mathematical expectation of any one of the number is, then, \$0.99021.

The matter will be still clearer if we examine it from another point of view. If 979 die within a year, there will be 99,021 survivors, each to receive a dollar, so that \$99,021 must be contributed. The amount that each man must contribute is therefore

$$\frac{\$99{,}021}{100{,}000} = \$0.99021.$$

While the notion of mathematical expectation had its origin at the gaming table, it has been made the basis of one of the great economic developments of modern times, viz. the development of the business of life insurance. The example just given shows that if a man forty years old wishes to provide \$10,000 for his estate in case his death should occur within one year, he would have to pay at least \$97.90 to the insurance company

agreeing to pay the amount to his estate, since the mathematical expectation would be the probability of the man's dying, viz. .00979 times the amount to be received. The cost would be greater than this figure, of course, since interest, cost of administering the business, and a fair profit to the company would have to be allowed.

Here, again, it should be carefully observed that it would be gambling, pure and simple, for an insurance company to promise, for a consideration of $125, to pay $10,000 to the estate of a single individual forty years of age in the event of the death of the individual within a year, though it would find the business not only perfectly safe but quite profitable if it could make 100,000 such contracts.

EXAMPLES

1. At a gaming table a stake of $100 is made contingent upon the event of a throw of a die being less than 4. What is the mathematical expectation of the player?

2. If 999 out of 1000 ships of a given class, and in a given condition, reach port safely, what would be the cost of insuring a ship belonging to the class in question, and its cargo, for $500,000 for a single voyage, if interest, expenses, and profits are neglected?

3. Experience has shown that, of 100,000 children aged 10 years, 749 die within 1 year. What should be the minimum cost of insuring the life of a 10-year-old child for $1000 for 1 year?

4. If it were shown that 2 out of every 1000 dwelling houses worth $5000 burn annually, what would be the amount of the "risk" assumed in insuring such a house for one year?

70. The mortality table. One of the most important applications of the theory of probability to human affairs is the application to problems having to do with the duration of human life. The greater number of such problems can be solved if we can answer the question, Given a certain number of persons all of the same age, how many will be alive at the end of each succeeding year until all are dead? The answer to this question, so far as it can be answered, must be obtained as a result of observation upon a large number of lives, and not by

any process of mathematics. The results of such observation carried on through a series of years are embodied in a table called a *mortality table.*

In its simplest form a mortality table is a table showing the number of persons, out of a large number all born on the same day, that survive at the end of each successive year until every individual of the group is dead. It would consist, therefore, of two columns: one a column of ages beginning with the age at which every member of the group is supposed to be alive; the other the number of persons living at each succeeding age. In practice at least three more columns, deduced from these two, are put into the mortality table: viz. one for the number of persons dying within each year; one for the probability that an individual will die between two consecutive ages; and one for the probability that an individual will live one year beyond any given age.

It is of course impossible to trace the careers of a very large number of children, say 100,000, all born on the same day, until all are dead, so that a mortality table cannot be constructed in this way. However, it is not at all necessary to do this. We are concerned with the probability that a person of a given age will live or die within a year, and, this being the case, the probabilities for two different ages may be determined wholly independently of each other. For example, the number dying within a year out of a group aged 40 and the number dying within a year out of a group aged 41 may be found from two wholly different groups. Indeed, it is not even necessary that the two groups be of the same size, provided both are large enough to furnish a safe basis upon which to work.

A mortality table is not only the foundation of all safe life-insurance business, but it is the basis of all computations having to do with life annuities and old-age pensions, and with many problems having to do with the administration of inheritance-tax laws. It will differ for different countries, for different periods in the same country, for different races living side by side in the same country, for persons of the same race engaged in different occupations. There are marked differences in tables constructed for males and for females.

The actual construction of a mortality table from observed data is a matter of considerable difficulty, and it would be out of place to attempt to follow out the work in detail here. The reader is referred for such information to standard textbooks on insurance. Many excellent tables have been constructed, some from the records of insurance companies, and others from vital statistics collected in various other ways.

Table XI, known as the American Experience Table, is based upon the records of the Mutual Life Insurance Company of New York, and was first published in 1868. It is the table most used in this country, and will be used in this book in all subsequent computations having to do with mortality statistics.

In what follows, the age of a person will be denoted by x, the number living at age x, out of the number with which the table starts, by l_x, and the number dying between ages x and $x+1$, by d_x. The probability that a person of age x will live at least one year is denoted by p_x, and the probability that a person of age x will die within the year, by q_x. The symbol (x) is frequently used to denote a person or a life aged x.

With the notation indicated above, and by means of the definition of probability, we have the following formulas:

$$d_x = l_x - l_{x+1}, \tag{1}$$

$$p_x = \frac{l_{x+1}}{l_x}, \tag{2}$$

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x}. \tag{3}$$

The probability that a person will live at least n years is the probability that he will be alive at the end of n years. It is denoted by ${}_np_x$. Therefore

$${}_np_x = \frac{l_{x+n}}{l_x}. \tag{4}$$

The probability that (x) will not live n years is denoted by $|_nq_x$. We have

$$|_nq_x = 1 - {}_np_x = \frac{l_x - l_{x+n}}{l_x}. \tag{5}$$

A few typical examples will suffice to show how to attack problems that may arise.

PROBLEM. *To find the probability that (x) and (y) will both survive n years.*

The survivals of the two lives are independent events, and the survival of both is the compound probability made up of the product of the two simple probabilities. Therefore, if ${}_np_{xy}$ denote the probability required,

$${}_np_{xy} = {}_np_x \cdot {}_np_y. \tag{6}$$

PROBLEM. *To find the probability that the life (x) will survive n years and the life (y) will fail within n years.*

Again we have to do with independent events, and consequently the required probability is

$${}_np_x \cdot {}_{\lvert n}q_y = {}_np_x(1 - {}_np_y).$$

PROBLEM. *To find the probability that at least one of two lives, (x) and (y), will survive n years.*

There are three mutually exclusive events, for each of which the *partial* probability is known. These partial probabilities are (1) the probability that both will survive, or ${}_np_{xy}$; (2) the probability that (x) will survive and (y) die, or ${}_np_x(1 - {}_np_y)$; (3) the probability that (x) will die and (y) survive, or $(1 - {}_np_x)\,{}_np_y$.

The *total* probability that at least one will survive n years is the sum of the three partial probabilities, or

$${}_np_x \cdot {}_np_y + {}_np_x(1 - {}_np_y) + (1 - {}_np_x)\,{}_np_y = {}_np_x + {}_np_y - {}_np_x \cdot {}_np_y.$$

EXAMPLES

1. Find the probability that a man 35 years of age will live to be 45.

Solution. By formula (4),

$${}_{10}p_{35} = \frac{l_{45}}{l_{35}},$$

and the table gives $l_{35} = 81{,}822$ and $l_{45} = 74{,}173$.

The required probability is therefore

$${}_{10}p_{35} = \frac{74{,}173}{81{,}822} = .90652.$$

2. A father is 40 years of age and his son is 10. What is the probability that both will be alive 11 years hence?

Solution. The probability that the father will live at least 11 years is

$$ {}_{11}p_{40} = \frac{68,842}{78,106} = .88139, $$

and the probability that the son will live 11 years is

$$ {}_{11}p_{10} = \frac{91,914}{100,000} = .91914. $$

The probability that both will live 11 years is the compound probability

$$ .88139 \times .91914 = .81012. $$

3. Find the probability that at least one of the two persons in Example 2 will survive 11 years.

Solution. The mutually exclusive events are as follows:

1. The survival of both.
2. The survival of the father and the death of the son.
3. The death of the father and the survival of the son.

The probability of the survival of both has been found in Example 2 to be .81021. The probability of the survival of the father and the death of the son is

$$ \begin{aligned} {}_{11}p_{40}(1 - {}_{11}p_{10}) &= .88139 \times (1 - .91914) \\ &= .07127. \end{aligned} $$

The probability of the survival of the son and the death of the father is

$$ \begin{aligned} {}_{11}p_{10}(1 - {}_{11}p_{40}) &= .91914 \times (1 - .88139) \\ &= .10902. \end{aligned} $$

The total probability that one of the two will survive is the sum of these partial probabilities, viz.

$$ .81012 + .07127 + .10902 = .99041. $$

This solution is an excellent illustration of the fact that in all cases it is better to work out the simplest possible formula before beginning the numerical computation. The formula of the last problem of the present section gives directly

$$ \begin{aligned} {}_{11}p_{40} + {}_{11}p_{10} - {}_{11}p_{40} \cdot {}_{11}p_{10} &= .88139 + .91914 - .81012 \\ &= .99041. \end{aligned} $$

4. A man aged 50 starts an enterprise requiring 10 years for its completion. What is the probability that he will live to see the end of the work?

5. A man and his wife are 28 and 26 years old, respectively, when their first child is born. What is the probability that both will live until the twenty-first anniversary of the child's birth?

6. Suppose that the man and wife and child in Example 5 are all living when the child is 10 years old. What is the probability that they will all be alive when the child is 21?

7. What is the probability that, of the two parents in Example 5, at least one will be alive on the twenty-first anniversary of the child's birth? that one and only one will be alive when the child is 21?

8. Under a certain pension system a man who had served 25 years in a given capacity was eligible to a retiring allowance. What would be the probability that under such a system a man beginning his service at the age of 30 would live to take advantage of a retiring allowance?

9. Find the probability that at least one of the two lives (x) and (y) will fail within n years?

10. Find the probability that (x) will live exactly n years, i.e. the probability that he will die between the ages $x + n$ and $x + n + 1$.

11. Approximately 2,800,000 men were mustered out of the two armies at the close of the Civil War in 1865. If we suppose that they were all 27 years of age, how many will probably be alive in 1915?

12. From the American Experience Table, plot the curve showing the number of deaths per 100,000 occurring annually from age 10 to the limit of the table.

13. Plot the curve showing the probability of dying for each year from the age of 10 according to the American Experience Table.

CHAPTER XII

LIFE ANNUITIES

71. Endowments and life annuities. A *life annuity* is an annuity the end of whose term is determined by the duration of one or more lives. If no confusion can arise, it is called simply an annuity. If the end of the term depends upon the duration of more than one life, the annuity is called a *joint life annuity.* The annual payments of a pension constitute a life annuity. Joint life annuities occur frequently in connection with inheritance-tax computations. For example, a father leaves an estate of \$100,000, the income of which is to go to his two sons as long as they shall both live. If the bequest is subject to an inheritance tax, the problem of determining its value to the two sons is at once in question. In most states the annual income would be estimated at 5% of the estate, so that the problem resolves itself into that of finding the present value of a joint life annuity of \$5000 per annum on a 5% basis.

A life annuity may be *due*, in which case the first payment is made immediately instead of at the end of one year; it may be *deferred*, in which case the first payment is made at the expiration of $n+1$ years, where n is the number of years the annuity is deferred; or it may be *temporary*, in which case the payments cease at the expiration of a given time, even though the annuitant be still alive. The payments of a deferred annuity will never begin should the annuitant die before the expiration of $n+1$ years. Nevertheless, the annuity has a definite present value.

The important thing to be determined in the case of life annuities is the present value of the annuity, and the present value can be obtained most easily through the notion of an *endowment*, or, as it is sometimes called, a *pure endowment.* An endowment is a sum payable to an individual, the *nominee*, at a

given future date, provided he survive to that date. The *present value of an endowment* of 1, payable at the end of n years to a person aged x, if he should live to the age $n+x$, is denoted by ${}_nE_x$.

PROBLEM. *To find the present value of an endowment of 1, payable after n years to a person aged x.*

First proof. If the sum of 1 were certain to be paid at the end of n years, the present value would be v^n; but since the payment is contingent upon the probability that the nominee will live to receive the sum of 1, the present value will be the mathematical expectation of receiving v^n contingent upon a known probability. The probability that a person aged x years will live at least n years is, by § 70,

$${}_np_x = \frac{l_{x+n}}{l_x}.$$

Therefore the present value of the endowment is

$${}_nE_x = v^n \cdot {}_np_x = v^n \cdot \frac{l_{x+n}}{l_x}. \tag{1}$$

In connection with this proof it is important to note that no equitable arrangement would be possible by which a single person aged x could pay the sum ${}_nE_x$ with the expectation of receiving 1 at the end of n years, for if the sum ${}_nE_x$ were put at interest for n years, the amount would be less than 1, and if the person were living, the amount would not be sufficient to redeem the promise. On the other hand, if the individual should die before n years, there would be at the end of n years the sum $(1+i)^n {}_nE_x$, with no one to claim it. The fact that such an arrangement could be made if participated in by a considerable number of persons is made clear in the alternative proof.

Second proof. Suppose that l_x persons, all of age x, enter into an agreement to contribute a sum sufficient to secure the payment of one dollar to each of the survivors at the end of n years. According to the mortality table the number of survivors would

be l_{x+n}, so that l_{x+n} dollars would be required at the end of the time. It follows that the sum required now would be $v^n l_{x+n}$, and the amount to be contributed to the fund by each one, if all share equally, is exactly

$$v^n \cdot \frac{l_{x+n}}{l_x},$$

which agrees with (1).

The present value of an endowment of a sum different from 1 is, of course, proportional to the amount to be received; for example, the endowment of \$100 payable in n years to a person aged x is

$$\$100\,{}_nE_x.$$

EXAMPLES

1. A father bequeathes to his son the sum of \$25,000, to be paid when the son is 21. If the father dies when the son is 15, what is the value, on a 5% basis, of the son's inheritance at the time of the father's death?

Solution. The probability that the son will live to receive the bequest is

$$\frac{l_{21}}{l_{15}} = \frac{91{,}914}{96{,}285} = .95460.$$

Furthermore, v^6 at 5% = .746215.

The present value of the bequest to the son is therefore

$$\$25{,}000 \times .746215 \times .95460 = \$17{,}808.42.$$

2. A father, dying when his son is 25, bequeathes to the son the sum of \$10,000, to be paid when the son is 30. What is the present value of the bequest on a 5% basis?

PROBLEM. *To find the present value of a life annuity of* 1 *per annum payable to a person aged x.*

Let a_x denote the present value of an annuity of 1 per annum payable to a person aged x. The several annual payments may be looked upon as so many endowments whose present values are

$${}_1E_x, \quad {}_2E_x, \quad {}_3E_x, \cdots.$$

Since each of these present values contains the probability of survival as a factor, the series will continue until the probability of survival becomes zero. It will continue, therefore, until $x + n$ in

the expression l_{x+n} reaches the highest age given in the mortality table, or, as we shall say for short, to table limit. We have, then,

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \cdots \text{ to table limit.} \tag{2}$$

If we replace each term of the series in (2) by its value in terms of l_x, l_{x+1}, $\cdots$, l_{x+n}, and powers of v, and note that l_x is a common denominator, we have

$$a_x = \frac{vl_{x+1} + v^2 l_{x+2} + \cdots \text{ to table limit}}{l_x}. \tag{3}$$

EXAMPLES

1. A pension of $500 per annum, payable at the end of each year, is granted to a man 92 years old. What is the present value of his estate, according to the American Experience Table, at $3\frac{1}{2}\%$?

Solution. By formula (3),

$$a_{92} = \frac{(1.035)^{-1} \cdot l_{93} + (1.035)^{-2} \cdot l_{94} + (1.035)^{-3} \cdot l_{95}}{l_{92}}.$$

The American Experience Table gives $l_{92} = 216$, $l_{93} = 79$, $l_{94} = 21$, and $l_{95} = 3$. Consequently,

$$a_{92} = \frac{(1.035)^{-1} \cdot 79 + (1.035)^{-2} \cdot 21 + (1.035)^{-3} \cdot 3}{216}$$
$$= .45666.$$

The value of the annuity of $500 per annum is, therefore,

$$\$500\, a_{92} = \$228.329.$$

2. Find the present value of the annuity in Example 1, on a 5% basis.

72. The computation of life annuities; commutation columns. Example 1 of the previous section, which was purposely chosen with the age of the annuitant near the limit of the table, shows us that the labor of computation for the younger ages would be very great. For example, if we wish to compute a_{21} by means of the American Experience Table, the numerator of (3), § 71, will be a series consisting of no less than 74 terms, each one of which must be computed separately. Fortunately, it is possible to abridge the computation greatly by means of what are known as *commutation columns.*

To get an idea of what is meant by commutation columns, let numerator and denominator of the right member of (3), § 71, be multiplied by v^x. The result is

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + \cdots \text{ to table limit}}{v^x l_x}. \tag{1}$$

Every term of both numerator and denominator of (1) is now of the same form, viz. $v^{x+r}l_{x+r}$.

If we define D_{x+r} by the equation

$$D_{x+r} = v^{x+r}l_{x+r}, \tag{2}$$

equation (1) will take the form

$$a_x = \frac{D_{x+1} + D_{x+2} + \cdots \text{ to table limit}}{D_x}. \tag{3}$$

Again, if we define N_{x+1} by the equation

$$N_{x+1} = D_{x+1} + D_{x+2} + \cdots \text{ to table limit}, \tag{4}$$

we have for a_x the very simple form

$$a_x = \frac{N_{x+1}}{D_x}. \tag{5}$$

We have also

$$1 + a_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + D_{x+1} + \cdots}{D_x} = \frac{N_x}{D_x}. \tag{6}$$

Tables giving the values of N and D for various ages, and at rates of interest in common use, have been computed and are available for use. Such tables are called *commutation tables* or *commutation columns* * (see Table XII). Other commutation columns will be described later on.

* The name *commutation columns* was given in 1840 by Professor De Morgan, who noticed that they formed a means of commuting, or exchanging, one benefit for another. To illustrate, the definitions of D_x and D_{x+n} enable us to write

$$\frac{D_{x+n}}{D_x} = \frac{v^{x+n}l_{x+n}}{v^x l_x};$$

whence

$$D_{x+n} = v^n D_x \cdot \frac{l_{x+n}}{l_x}$$

= present value of D_x due n years hence into the probability that a person aged x will survive n years.

In words, one could give up D_{x+n} now for D_x payable n years hence. The other quantities, N_x, C_x, and M_x, have similar properties.

By means of the proper commutation columns the computation of life annuities becomes a matter of simple division. Great care must be taken, however, to use columns based upon the mortality table and the rate of interest that should be used in the problem in question. For example, at the present time, in Wisconsin, the American Experience Table is the standard and 5% interest is specified by law for all inheritance-tax computations in which annuities are involved; on the other hand, the law of Massachusetts specifies that for similar computations a table known as the Actuaries', or Combined Experience, Table and 4% must be used. Still other tables are used in other states. Table No. XII is the American Experience Table with interest assumed at $3\frac{1}{2}\%$. This table and interest at $3\frac{1}{2}\%$ is a combination widely used for insurance work in this country. It will be used exclusively in this book.

EXAMPLES

1. What is the present value of a life annuity of $500 per annum for a person aged 21 years?

Solution. By formula (5), $a_{21} = \frac{N_{21}}{D_{21}}$

$$= \frac{893{,}212.5}{44{,}630.8}$$

$$= 20.013365.$$

The annuity of $500 has a present value of

$$\$500 \times 20.013365 = \$10{,}006.68.$$

2. A man left an estate worth approximately $2,000,000, and in his will directed that the income of the estate should go to his widow, aged 60 years at the time of his death, for the remainder of her life. Suppose the law specifies that the annual income shall be considered as 5% of the estate, and that an inheritance tax of $1\frac{1}{2}\%$ must be paid. What will be the amount of the inheritance tax upon the wife's interest?

3. Express ${}_nE_x$ in terms of D's.

73. Deferred and temporary life annuities.

PROBLEM. *To find the present value of a life annuity payable to a person aged x after the lapse of n years.*

By the definition of a deferred annuity the first payment is to be made at the end of $n+1$ years, and the payments will

continue until the life fails. The present value will be the sum of the present values of a series of endowments due $n+1$, $n+2, \cdots$, years, to the table limit. Denoting the present value of the annuity by ${}_{n|}a_x$, we have

$$ {}_{n|}a_x = {}_{n+1}E_x + {}_{n+2}E_x + {}_{n+3}E_x + \cdots \text{ to table limit,} \tag{1}$$

or, replacing each one of the E's by their values as given by (1), § 71, and taking out the common denominator l_x,

$$ {}_{n|}a_x = \frac{v^{n+1}l_{x+n+1} + v^{n+2}l_{x+n+2} + \cdots \text{ to table limit}}{l_x}. \tag{2}$$

To express ${}_{n|}a_x$ in terms of the commutation symbols D and N, we multiply numerator and denominator by v^x. The result is, as can easily be seen,

$$ {}_{n|}a_x = \frac{N_{x+n+1}}{D_x}, \tag{3}$$

where $n+1$ is the number of years to elapse before the first payment is made.

PROBLEM. *To find the present value of a temporary annuity.*

A temporary annuity has been defined as one whose first payment begins one year hence and continues n years, provided the annuitant should live so long. It consists of n payments contingent upon the continued life of the annuitant. Its present value is denoted by the symbol $|_n a_x$, where x is the age of the annuitant.

To solve the problem, note that a life annuity may be considered as a temporary annuity for n years, followed by a life annuity deferred n years. The equation

$$ a_x = {}_{n|}a_x + |_n a_x \tag{4}$$

is, therefore, identically true, so that the value of the temporary annuity is given by the formula

$$ |_n a_x = a_x - {}_{n|}a_x. \tag{5}$$

By means of (5) of § 72 and (3) of the present section, (5) is reduced to the form

$$ |_n a_x = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \tag{6}$$

EXAMPLES

1. A father bequeathes to his son an annuity of \$500, to begin when the son is 40. What is the value of the son's interest at the time of the father's death, which occurs when the son is 30?

Solution. Formula (3) gives

$$
\begin{aligned}
{}_{10|}a_{30} &= \frac{N_{41}}{D_{30}} \\
&= \frac{324{,}440.0}{30{,}440.8} = 10.65803.
\end{aligned}
$$

The deferred annuity of \$500 per annum has a present value of

$$\$500 \times 10.65803 = \$5829.02.$$

2. What is the present value of a temporary annuity of \$500 per annum for 10 years, payable to a person whose age is 21?

Solution. By formula (6),

$$
\begin{aligned}
{}_{|10}a_{21} &= \frac{N_{22} - N_{32}}{D_{21}} \\
&= \frac{893{,}212.5 - 537{,}199.3}{44{,}630.8} \\
&= 7.976850
\end{aligned}
$$

The present value of the annuity of \$500 per annum under similar circumstances is, therefore,

$$\$500 \times 7.976850 = \$3988.42.$$

3. The Carnegie Foundation for the Advancement of Teaching grants, as a retiring allowance to a professor who has reached the age of 65, an annual stipend equal in amount to one half the professor's active pay plus \$400. What is the privilege worth to a professor aged 50 years whose active pay is \$3000?

4. Find the value of ${}_{n|}a_x$ by using the notion of the endowment.

5. Find the value of ${}_{|n}a_x$ by using the notion of the endowment.

CHAPTER XIII

SOME PROBLEMS IN LIFE INSURANCE

74. Definitions. An *insurance*, in its broadest sense, is an indemnity against loss. The business of furnishing insurance is conducted by a corporation or an association. It consists, primarily, in collecting sums of money from those who wish themselves or their heirs to be protected against loss incurred through the happening of a stipulated event, and distributing the funds thus collected to those to whom the stipulated event has happened, or to their heirs. In life insurance the stipulated event is the death of the insured.

The written agreement, or contract, between the insured person and the corporation or association furnishing the insurance is called a *policy*, and the insured person is called a *policyholder*. The amount received under the terms of the policy is called a *benefit*, and the person to whom it is paid is called the *beneficiary*.

Among the many forms of life insurance in existence we shall consider only the simpler forms furnished by the so-called *legal-reserve*, or *old-line*, insurance companies. A legal-reserve life insurance company offers insurance for a specified sum, called the *gross*, or *office premium*, or simply the *premium*. The gross premium may be defined as "the amount paid, or agreed to be paid, in one sum, or periodically, for a contract of insurance." Whether the premium is paid in a lump sum or in annual installments, it is invariably paid in advance. The period of twelve calendar months, reckoned from the date on which the policy goes into effect, or from any anniversary of this date, is called the *policy year*.

The first problem that confronts the life insurance company is the determination of the amount to be paid by the policyholder. The two elements having most to do with this determination are,

first, the *death rate*, and second, the *rate of interest that can be realized on investments.*

The death rate is defined as the number of deaths that will occur in a definite group of individuals of given age and class in one year. It is determined by means of a mortality table, such as has been considered in § 70, and, for insurance purposes, is expressed as the probability that a person of given age will die within one year.

The kinds of life insurance offered by life insurance companies may be divided, for convenience, into two classes, viz. *whole life insurance* and *term insurance.*

A whole life insurance is an insurance in which the company agrees to pay the benefit on satisfactory proofs of the policyholder's death. A term insurance is one in which the benefit is payable on the death of the insured, provided death should occur within a specified term of years.

The first problem in the mathematics of life insurance, after one has constructed a mortality table, is the determination of what is called the *net premium.* The net premium is the sum that would be paid by the insured as the equivalent of the benefit guaranteed by the company under the following conditions:

(1) That the death rate is exactly the rate given by the mortality table adopted as the standard.

(2) That the benefit is payable at the end of the policy year in which death occurs.

(3) That the interest rate is exactly the assumed rate.

(4) That the business is conducted without expense.

The net premium, when expressed as a single sum, is called the *net single premium.* The net single premium is, under the assumptions made, the mathematical equivalent of the benefit; i.e. it is the present value of the benefit.

The solutions of the problems in life insurance are very similar to the solutions of the problems of life annuity, the difference being that the life annuity has to do with the probability of living, while the life insurance has to do, for the most part, with the probability of dying. For the sake of simplicity the benefit is assumed to be 1 in all computations.

75. Whole life insurance.

PROBLEM. *To find the net single premium for a whole life insurance of* 1 *on the life of a person aged x years.*

Suppose l_x persons, all of age x, agree to pay to a company a sum sufficient to secure to the estate of each one the payment of \$1, payable at the end of the policy year in which death occurs. The number of persons dying during the successive years will be, according to the mortality table,

$$d_x, \quad d_{x+1}, \quad d_{x+2}, \quad \text{to table limit.}$$

The sum d_x required at the end of the first year would have a present value of vd_x. Likewise, the sums required at the end of each of the succeeding years would have for their present values

$$v^2d_{x+1}, \quad v^3d_{x+2}, \cdots, \text{ to table limit.}$$

The total amount required will be, therefore,

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to table limit.}$$

If this amount be shared equally among the l_x persons, the amount paid by each one will be the net single premium, or A_x. Therefore

$$A_x = \frac{vd_x + v^2d_{x+1} + \cdots \text{ to table limit}}{l_x}. \tag{1}$$

The computation required to find A_x from equation (1) is quite simple, though it would be very long for the younger ages. To adapt (1) for computation, we make use of the same device that was used in computing a_x in § 72. Multiplying numerator and denominator by v^x, we note that the denominator is changed to D_x, defined by equation (2) in § 72. If, moreover, we define C_x by the equation

$$C_x = v^{x+1}d_x, \tag{2}$$

equation (1) will take the form

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \cdots \text{ to table limit}}{D_x}. \tag{3}$$

It is customary to denote the sum of the terms in the numerator by M_x, so that, by definition,

$$M_x = C_x + C_{x+1} + C_{x+2} + \cdots \text{ to table limit.} \tag{4}$$

With this notation A_x takes the simple form

$$A_x = \frac{M_x}{D_x}. \tag{5}$$

The expressions C_x and M_x, which possess the same peculiar property as the expressions D_x and N_x of § 72, are usually tabulated along with the commutation columns for D_x and N_x (see Table XII).

EXAMPLES

1. What is the net single premium on a whole life policy for $10,000 on the life of a person aged 35?

Solution. By formula (5),
$$\begin{aligned} A_{35} &= \frac{M_{35}}{D_{35}} \\ &= \frac{9094.955}{24,544.7} \\ &= .370547. \end{aligned}$$

The net single premium for $10,000 is, therefore,

$$\$10,000 \times A_{35} = \$3705.47$$

2. Find the net single premium for a whole life policy for $10,000 on a life aged 30 years.

When the premium is paid in annual installments, the annual payments may continue throughout the life of the policyholder, or they may continue for a definite term of years. A whole life policy on which the payments continue through life is called an *ordinary life policy*, while a whole life policy on which the payments continue for a stated number of years is called a *limited payment policy*, or an *n-payment life policy*, where n denotes the number of annual payments that are to be made unless death should occur earlier.

The sum which, if paid at the beginning of each policy year for a term of years, or for life, would make the equivalent of the net single premium is called the *net annual premium*.

PROBLEM. *To find the net annual premium for an ordinary life policy on the life of a person aged x.*

By the definitions the net annual premiums for an ordinary life policy constitute a life annuity which, since the premiums

are paid in advance, is *due.* Let P_x denote the net annual premium. The present value of the annuity is, therefore,

$$P_x + P_x \cdot a_x,$$

which must equal the net single premium. Consequently, we have the equation

$$P_x + P_x \cdot a_x = A_x,$$

from which we find
$$P_x = \frac{A_x}{1 + a_x}. \tag{6}$$

If we express A_x and a_x in terms of commutation symbols according to (6) of § 72 and (5) of the present section, we have, after reduction,
$$\boldsymbol{P_x = \frac{M_x}{N_x}}. \tag{7}$$

PROBLEM. *To find the net annual premium for an n-payment life policy on the life of a person aged x.*

The net annual premiums for an n-payment life policy constitute a cash payment and a temporary annuity for $n-1$ years, unless death should occur earlier. If ${}_nP_x$ denote the net annual premium, we have
$${}_nP_x + {}_nP_x \cdot |_{n-1}a_x = A_x,$$

from which we find
$${}_nP_x = \frac{A_x}{1 + |_{n-1}a_x}. \tag{8}$$

In commutation symbols, (8) has the form

$$\boldsymbol{{}_nP_x = \frac{M_x}{D_x + N_{x+1} - N_{x+n}} = \frac{M_x}{N_x - N_{x+n}}}. \tag{9}$$

EXAMPLES

1. Find the net annual premium for an ordinary life policy of $10,000 on a life aged 27 years?

Solution. The net annual premium is given by formula (7), so that we have

$$P_{27} = \frac{M_{27}}{N_{27}}$$
$$= \frac{11{,}053.97}{696{,}333.8} = .0158745.$$

The net annual premium on an ordinary life policy for $10,000 would be

$$\$10{,}000 \times .0158745 = \$158.745.$$

2. Find the net annual premium for a twenty-payment life policy of \$10,000 on the life of a person aged 25 years.

3. Find the expression for A_x by using the probability of death for each age in the mortality table, instead of the number of persons dying out of the group l_x.

4. Prove that $M_x = vN_x - N_{x+1}$.

76. Term insurance.

PROBLEM. *To find the net single premium for a term insurance of 1 for n years on the life of a person aged x.*

The probability that a person aged x will die within the first year is $\frac{d_x}{l_x}$, and the present value of the mathematical expectation that the benefit will have to be paid is $v\frac{d_x}{l_x}$. Similarly, the probability that the person will die within the second year is $\frac{d_{x+1}}{l_x}$, and the present value of the mathematical expectation that the benefit will have to be paid at the end of the second year is $v^2\frac{d_{x+1}}{l_x}$, and so on for n years. The present value of the mathematical expectation for the nth year will be $v^n\frac{d_{x+n-1}}{l_x}$. Denoting the net single premium by $|_nA_x$, we have

$$|_nA_x = \frac{vd_x + v^2d_{x+1} + \cdots + v^nd_{x+n-1}}{l_x}. \tag{1}$$

If numerator and denominator of (1) be multiplied by v^x, and note be taken of the fact that

$$v^{x+r+1}d_{x+r} = C_{x+r},$$

equation (1) becomes

$$|_nA_x = \frac{C_x + C_{x+1} + \cdots + C_{x+n-1}}{D_x}. \tag{2}$$

The numerator of (2) is the numerator of (3) of § 75, or, what amounts to the same thing, of (5) of § 75, less the terms of the sum

$$C_{x+n} + C_{x+n+1} + \cdots \text{ to table limit.} \tag{3}$$

The numerator of (5), § 75, is M_x, while the sum of the terms of (3) is

$$M_{x+n} = C_{x+n} + C_{x+n+1} + \cdots \text{ to table limit.}$$

Therefore equation (2) is reduced to the final form,

$$|_n A_x = \frac{M_x - M_{x+n}}{D_x}. \tag{4}$$

EXAMPLE

Find the net single premium for a term insurance of $10,000 for 10 years on the life of a person aged 46.

Solution. By formula (4),

$$|_{10}A_{46} = \frac{M_{46} - M_{56}}{D_{46}}$$

$$= \frac{7022.682 - 5335.898}{15,070.0} = .111930.$$

The net single premium is, therefore,

$$\$10,000 \times .111930 = \$1119.30.$$

PROBLEM. *To find the net annual premium for a term insurance of* 1 *for n years on the life of a person aged x years.*

The net annual premiums will constitute a temporary life annuity which is due, since the premiums are payable in advance. Let the net annual premium be denoted by $|_n P_x$. The present value of the annuity will therefore be

$$|_n P_x + |_n P_x \cdot |_{n-1}a_x,$$

which must be equal to the net single premium $|_n A_x$. The unknown quantity $|_n P_x$ is therefore determined by the equation

$$|_n P_x + |_n P_x \cdot |_{n-1}a_x = |_n A_x.$$

Consquently,

$$|_n P_x = \frac{|_n A_x}{1 + |_{n-1}a_x}. \tag{5}$$

If we substitute for $|_{n-1}a_x$ and $|_n A_x$ their values in terms of D's, N's, and M's as given by (6) of § 73 and (4) of the present section, we find

$$|_n P_x = \frac{M_x - M_{x+n}}{D_x + N_{x+1} - N_{x+n}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \tag{6}$$

EXAMPLES

1. Find the net annual premium for a term insurance of \$10,000 for 10 years on the life of a person aged 46 years.

Solution. Formula (6) gives

$$\begin{aligned} |_{10}P_{46} &= \frac{M_{46} - M_{56}}{N_{46} - N_{56}} \\ &= \frac{7022.682 - 5335.898}{237{,}971.9 - 115{,}142.4} \\ &= \frac{1686.784}{122{,}829.5} \\ &= .0137327. \end{aligned}$$

The net annual premium on a policy for \$10,000 under the same conditions is, therefore,

$$\$0.0137327 \times 10{,}000 = \$137.327.$$

2. Find the net annual premium on a policy of \$10,000 for 10 years on the life of a person aged 42 years.

3. Find the expression for $|_nP_x$ in terms of N's alone.

77. Endowment insurance. An *endowment insurance* is a form of insurance in which the company agrees to pay a given sum in the event either of the death of the policyholder or of his survival to the end of a specified term of years.

PROBLEM. *To find the net single premium for an endowment insurance of* 1 *on the life of a person aged x years.*

An endowment insurance may be looked upon as a term insurance for n years, together with a pure endowment payable at the end of n years in case the life survives. American writers use the symbol ${}_e\Pi_{x\,n}$ to denote the net single premium of an endowment assurance. We have, therefore,

$${}_e\Pi_{x\,n} = |_nA_x + {}_nE_x. \qquad (1)$$

But, by (4), § 76,

$$|_nA_x = \frac{M_x - M_{x+n}}{D_x}.$$

By (1), § 71, and (2), § 72,

$${}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v_x l_x} = \frac{D_{x+n}}{D_x}.$$

Substituting these values in (1), we obtain

$$ {}_e\Pi_{x\,n} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{2} $$

PROBLEM. *To find the net annual premium for an endowment insurance of* 1 *for n years on the life of a person aged x years.*

Let ${}_eP_{x\,n}$ denote the net annual premium. The first installment is a cash payment of ${}_eP_{x\,n}$, and the remaining payments constitute a temporary annuity of ${}_eP_{x\,n}$ for $n-1$ years. To determine ${}_eP_{x\,n}$, we have the equation

$$ {}_eP_{x\,n} + {}_eP_{x\,n} \cdot |_{n-1}a_x = {}_e\Pi_{x\,n}, \tag{3} $$

and, consequently,

$$ {}_eP_{x\,n} = \frac{{}_e\Pi_{x\,n}}{1 + |_{n-1}a_x}. \tag{4} $$

By (2) of the present section and (6) of § 73, equation (4) is reduced to the final form,

$$ {}_eP_{x\,n} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \tag{5} $$

EXAMPLES

1. Compute the net annual premium on a thirty-year endowment insurance of $1000 on the life of a person aged 24 years.

Solution. By formula (5)

$$ \begin{aligned} {}_eP_{24\;30} &= \frac{D_{54} + M_{24} - M_{54}}{N_{24} - N_{54}} \\ &= \frac{10{,}252.4 + 11{,}935.38 - 5682.861}{809{,}420.6 - 135{,}128.2} \\ &= \frac{16{,}504.92}{674{,}292.4} \\ &= .024477. \end{aligned} $$

The premium on a policy for $1000, under the same conditions, would be $24.48.

2. Compute the net single premium on a thirty-year endowment policy for $1000 on a life aged 24 years.

3. Express ${}_eP_{x\,n}$ in terms of N's, D's, and v.

78. Reserves; valuation of policies. Complete mortality tables show that from the age of ten or twelve years onward the death rate increases. If, therefore, the annual premium is a *level premium*, i.e. a premium which remains constant throughout the whole time that the policy remains in force, it will be higher than the net annual premium for a one-year insurance during the earlier years and lower than for a one-year insurance during the later years. For example, the probability that a person aged twenty-five years will die within one year is, by the American Experience Table, .008065. Consequently, on a $3\frac{1}{2}\%$ basis the net single premium on a one-year policy for \$1000 would be

$$\frac{\$0.008065}{1.035} \times 1000 = \$7.79.$$

Likewise, the premium for a one-year insurance on a policy for \$1000 at age sixty would be \$25.80. The net level premium on an ordinary life policy for \$1000 purchased at age twenty-five is \$15.10. The excess paid during the earlier years is an overpayment on the part of the policyholder and constitutes a reserve to meet the deficiency that will occur in later years.

The total excess accumulated to the end of a given policy year at the assumed rate of interest, and on the assumption that the death rate is exactly the tabular rate, is called the *terminal reserve* for that year and is known as the *value of the policy* or the *net value of the policy* at the end of the year in question.

To view the matter from another standpoint, let us examine an ordinary life policy for \$1000 on a life of age thirty. According to the American Experience Table the net annual premium at $3\frac{1}{2}\%$ is \$17.19. The number living at age thirty, out of 100,000 who were alive at age ten, is 85,441. Assuming that 85,441 policies of \$1000 are issued at the same time on 85,441 lives, all aged thirty years, the amount of the premiums received will be \$1,468,730.79, which, at $3\frac{1}{2}\%$, will amount in one year to \$1,520,136.37. According to the table 720 persons will die within the year, so that the death losses will be \$720,000. This will leave with the company a total of \$800,136.37, which would enable it to place \$9.45 to the credit of each of the 84,721

survivors. At the beginning of the second year there will be in the hands of the company the reserve fund of \$800,136.37, together with the premiums paid by the 84,721 survivors, each paying \$17.19, or a total of \$2,256,490.36. Beginning with this amount the computation for the reserve for the second year will be exactly like that for the first year. The following table will explain itself.

Table showing terminal reserves on an ordinary life policy for \$1000 *on the life of a person aged* 30 *years.*

Age of policy in years	Funds on hand at beginning of year	Fund accumulated at $3\frac{1}{2}\%$	Death losses	Fund at end of year	Amount to credit of each survivor
1	\$1,468,730.79	\$1,520,136.37	\$720,000	\$800,136.37	\$9.45
2	2,256,490.36	2,335,467.52	721,000	1,614,467.52	19.22
3	3,058,427.52	3,165,472.48	723,000	2,442,472.48	29.33
4	3,874,004.11	4,009,594.25	726,000	3,283,594.25	39.78
...	...	...	...	...	...

The amounts in the last column are the terminal reserves required. If the table were continued to the limit of the mortality table, the terminal reserve for the sixty-fifth year of the age of the policy, i.e. the ninety-fifth year of the age of the policyholder, would be exactly \$1000. Such reserves exist for all level premium policies covering a period longer than one year.

PROBLEM. *To find the terminal reserve for an ordinary life policy in general terms.*

We note that, at the end of any given year after the policy has been issued, the sum of the terminal reserve and the present value of the premiums yet unpaid will be equal to the net single premium for a new policy on the life of the insured person at the age attained. Let n denote the number of years that have elapsed since the policy was issued, and ${}_nV_x$ the reserve at the end of n years on a policy issued on a life aged x years. The age of the policyholder on the date in question will be $x+n$ years, and by § 75 the present value of the unpaid premiums would be

$$P_x(1+a_{x+n}),$$

while the net single premium for a new policy at the attained age, $x+n$, would be

$$A_{x+n}.$$

Consequently, the equation for the determination of ${}_nV_x$ is

$$A_{x+n} = {}_nV_x + P_x(1 + a_{x+n}). \tag{1}$$

From equation (1),

$${}_nV_x = A_{x+n} - P_x(1 + a_{x+n}). \tag{2}$$

The known quantities on the right may be easily expressed in terms of commutation symbols, for, by (6) of § 72, and (5) and (7) of § 75, we have

$$1 + a_{x+n} = \frac{N_{x+n}}{D_{x+n}},$$

$$A_{x+n} = \frac{M_{x+n}}{D_{x+n}},$$

$$P_x = \frac{M_x}{N_x}.$$

Substituting these values in (2), we have

$${}_nV_x = \frac{M_{x+n}}{D_{x+n}} - \frac{M_x}{N_x} \cdot \frac{N_{x+n}}{D_{x+n}}, \tag{3}$$

or

$${}_nV_x = \frac{N_x M_{x+n} - M_x N_{x+n}}{N_x \cdot D_{x+n}}. \tag{4}$$

Formulas similar to (2), (3), and (4) may be found for other forms of policies.

The question of reserves is one of very great importance in determining the condition of an insurance company. Indeed, the maintenance of a fund sufficient to cover the terminal reserves on all policies in force has been made the test of solvency and in many states is enforced by law.

The reserve is also the basis upon which the so-called *loan value*, or the *cash surrender value*, is determined. The cash surrender value is the amount returned to the policyholder in case the policy is allowed to lapse. It is usually equal to the loan value, or the amount which the company will loan with the policy as collateral security.

For various reasons it would not be fair to the company to give to the policyholder who withdraws the full amount of the reserve on his policy, at least during the earlier years of the life of the policy. The most important reason is the fact that it costs much more to get "new business" than to carry business already on the company's books. The insuring company is therefore allowed a *surrender charge* if the policyholder asks for the value of his policy either in cash or in extended insurance. The amount of the surrender charge is in a measure arbitrary and in most states is determined by law. It varies from $2\frac{1}{2}\%$ to 3% of the amount insured.

EXAMPLES

1. A man takes out an ordinary life policy for $1000 at age 30. When he is 50 years of age, the company decides to go out of business. What sum is due him?

Solution. By formula (4),

$$\begin{aligned} {}_{20}V_{30} &= \frac{M_{50}N_{30} - M_{30}N_{50}}{N_{30} \cdot D_{50}} \\ &= \frac{6{,}335.436 \times 596{,}803.6 - 10{,}259.02 \times 181{,}663.4}{596{,}803.6 \times 12{,}498.6} \\ &= .25704. \end{aligned}$$

The terminal reserve on the policy of $1000 would then be $257.04.

2. Find the terminal reserve on the above policy 60 years after it is issued, provided the holder is still alive.

3. Verify the result obtained in Example 2 by carrying out a table similar to that on page 210 and using $17.19 as the net annual premium on $1000 at age 30.

Suggestion. The reserve should be large enough to meet the demand that, according to the mortality table, will surely be made upon it at the end of the sixty-fifth year.

79. Loading; gross premiums. An insurance company could not do business on the basis of net premiums alone, for the net premium is found on the supposition that the business is conducted without expense. The most important items of expense in connection with the business are

1. The expenses incurred in getting new business, such as the agent's commission, which takes a large part of the first annual premium, cost of medical examination, and so on.

2. The cost of administering the business, such as salaries of administrative officers, cost of collecting premiums, and so on.

3. A small amount to cover unforeseen contingencies.

To meet these expenses the net premium is increased, sometimes by a percentage of itself, sometimes by a fixed charge, and sometimes by both. The increase is called *loading*. The sum of the net premium and the loading is called the *gross* or *office premium*. The gross premium is the amount actually paid by the policyholder.

It is customary to denote the gross premium by the same symbol that is used to denote the net premium, but with an accent. Thus, for an ordinary life policy P_x denotes the net annual premium and P'_x the gross annual premium. If k denote the rate at which the percentage charge is made, and c the fixed charge, the formula for P'_x in terms of P_x would be

$$P'_x = P_x(1+k) + c. \tag{1}$$

If the whole of the loading is considered as a fixed charge, k will be zero, while if it is looked upon as a percentage charge, c will be zero.

It would seem to be more in accord with scientific method if one could determine the sum total of the loading and then distribute it as an annual charge throughout the life of the policy. For example, suppose that the initial expenses incurred in connection with an ordinary life policy amount to K, and that all other expenses are met by an annual charge c. To distribute the amount K throughout the life of the policy, it must be looked upon as the present value of a life annuity. The annual rent R will then be given by the formula

$$K = R(1 + a_x), \tag{2}$$

and we have

$$R = \frac{K}{1+a_x}. \tag{3}$$

The total loading would then be $R + c$, and the gross premium would be

$$P'_x = P_x + R + c,$$

or

$$P'_x = P_x + \frac{K}{1+a_x} + c. \tag{4}$$

Similar expressions may easily be obtained for the gross annual premiums on other forms of policies. It is scarcely worth while to spend time in deriving formulas which are not in agreement with current practice, however faulty such practice may be, except in so far as they may enable the student to discuss intelligently propositions looking toward better practice.

EXAMPLES

1. Derive the formula for the gross annual premium on a twenty-payment life policy on the supposition that the initial expense is to be distributed as an annual charge to be made so long as the payments continue.

2. Express formula (4) above in terms of commutation symbols, so far as is possible.

3. Find the gross premium in terms of commutation symbols on an n-year endowment policy when the initial expense is distributed as a constant annual charge during the life of the policy.

80. Concluding note on life insurance. In the foregoing sections an attempt has been made to present in the barest outline a few of the more important problems that present themselves in the theory of life insurance. It may not be amiss to call attention to some of the subjects that the student of life insurance should take up.

1. In the first place, a more extended study of reserves will be indispensable.

2. No mention has been made of the "*surplus*." The net premium has been computed on the assumptions (1) that the death rate is exactly that of the mortality table in use; (2) that the interest rate is exactly the assumed rate; (3) that no policies will be allowed to lapse before the death of the policyholder or the expiration of the term for which the insurance was written. Moreover, the loading required to pay all expenses was supposed to be known in advance. Sound business policy requires that every one of these assumptions shall be made conservatively. The careful actuary will therefore adopt a mortality table that will surely cover the mortality. The loading should be made large enough to provide for every possible contingency that may arise.

It is clear, then, that at the end of each year a surplus may remain after all expenses have been met and sufficient provision has been made for the maintenance of adequate reserves for all policies in force. If the company is a mutual company, this surplus belongs to the policyholders, and an equitable method for distributing it must be found.

3. Many policies are written, under the terms of which the benefit is payable on the failure of one or both of two lives. Such a policy is called a joint life policy. The theory of joint life insurance is not essentially different from insurances on single lives, though many modifications must be made and new commutation columns must be constructed.

PART IV. TABLES

TABLE I.—THE NUMBER OF EACH DAY OF THE YEAR COUNTING FROM JANUARY 1

Day of Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29		88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

Note.—For leap years the number of the day is one greater than the tabular number after February 28.

TABLE II.—EXACT INTEREST AT 5% FOR TIMES FROM 1 TO 365 DAYS

$$I = Pi \frac{d}{365}$$

Days	$1,000	$2,000	$3,000	$4,000	$5,000	$6,000	$7,000	$8,000	$9,000	Days
1	0.1370	0.2740	0.4110	0.5749	0.6849	0.8219	0.9589	1.0959	1.2329	1
2	0.2740	0.5479	0.8219	1.0959	1.3699	1.6438	1.9178	2.1918	2.4658	2
3	0.4110	0.8219	1.2329	1.6438	2.0548	2.4657	2.8767	3.2877	3.6986	3
4	0.5479	1.0959	1.6438	2.1918	2.7397	3.2877	3.8356	4.3836	4.9315	4
5	0.6849	1.3699	2.0548	2.7397	3.4247	4.1096	4.7945	5.4795	6.1644	5
6	0.8219	1.6438	2.4658	3.2877	4.1096	4.9315	5.7534	6.5753	7.3973	6
7	0.9589	1.9178	2.8767	3.8356	4.7945	5.7534	6.7123	7.6712	8.6301	7
8	1.0959	2.1918	3.2877	4.3836	5.4795	6.5753	7.6712	8.7671	9.8630	8
9	1.2329	2.4658	3.6986	4.9315	6.1644	7.3973	8.6301	9.8630	11.0959	9
10	1.3699	2.7397	4.1096	5.4795	6.8493	8.2192	9.5890	10.9589	12.3288	10
20	2.7397	5.4795	8.2192	10.9589	13.6986	16.4384	19.1781	21.9178	24.6575	20
30	4.1096	8.2192	12.3288	16.4384	20.5479	24.6575	28.7671	32.8767	36.9863	30
40	5.4795	10.9589	16.4384	21.9178	27.3973	32.8767	38.3562	43.8356	49.3151	40
50	6.8493	13.6986	20.5479	27.3973	34.2466	41.0959	47.9452	54.7945	61.6438	50
60	8.2192	16.4384	24.6575	32.8767	41.0959	49.3151	57.5342	65.7534	73.9726	60
70	9.5890	19.1781	28.7671	38.3562	47.9452	57.5342	67.1233	76.7123	86.3014	70
80	10.9589	21.9178	32.8767	43.8356	54.7945	65.7334	76.7123	87.6712	98.6301	80
90	12.3288	24.6575	36.9863	49.3151	61.6438	73.9726	86.3014	98.6301	110.9589	90
100	13.6986	27.3973	41.0959	54.7945	68.4931	82.1918	95.8904	109.5890	123.2877	100
110	15.0685	30.1370	45.2055	60.2740	75.3425	90.4110	105.4795	120.5479	135.6164	110
120	16.4384	32.8767	49.3151	65.7534	82.1918	98.6301	115.0685	131.5068	147.9452	120
130	17.8082	35.6164	53.4247	71.2329	89.0411	106.8493	124.6575	142.4658	160.2740	130
140	19.1781	38.3562	57.5342	76.7123	95.8904	115.0685	134.2466	153.4247	172.6027	140
150	20.5479	41.0959	61.6438	82.1918	102.7397	123.2877	143.8356	164.3836	184.9315	150
160	21.9178	43.8356	65.7534	87.6712	109.5890	131.5068	153.4247	175.3425	197.2603	160
170	23.2877	46.5753	69.8630	93.1507	116.4384	139.7260	163.0137	186.3014	209.5890	170
180	24.6575	49.3151	73.9726	98.6301	123.2877	147.9452	172.6027	197.2603	221.9178	180
190	26.0274	52.0548	78.0822	104.1096	130.1370	156.1644	182.1918	208.2192	234.2466	190
200	27.3973	54.7945	82.1918	109.5890	136.9863	164.3836	191.7808	219.1781	246.5753	200
210	28.7671	57.5342	86.3014	115.0685	143.8356	172.6027	201.3699	230.1370	258.9041	210
220	30.1370	60.2740	90.4110	120.5479	150.6849	180.8219	210.9589	241.0959	271.2329	220
230	31.5068	63.0137	94.5205	126.0274	157.5342	189.0411	220.5479	252.0548	283.5616	230
240	32.8767	65.7534	98.6301	131.5068	164.3836	197.2603	230.1370	263.0137	295.8904	240
250	34.2466	68.4932	102.7397	136.9863	171.2329	205.4795	239.7260	273.9726	308.2192	250
260	35.6164	71.2329	106.8493	142.4658	178.0822	213.6986	249.3151	284.9315	320.5479	260
270	36.9863	73.9726	110.9589	147.9452	184.9315	221.9178	258.9041	295.8904	332.8767	270
280	38.3562	76.7123	115.0685	153.4247	191.7808	230.1370	268.4932	306.8493	345.2055	280
290	39.7260	79.4521	119.1781	158.9041	198.6301	238.3562	278.0822	317.8082	357.5342	290
300	41.0959	82.1918	123.2877	164.3836	205.4795	246.5753	287.6712	328.7671	369.8630	300
310	42.4658	84.9315	127.3973	169.8630	212.3288	254.7945	297.2603	339.7260	382.1918	310
320	43.8356	87.6712	131.5068	175.3425	219.1781	263.0137	306.8493	350.6849	394.5205	320
330	45.2055	90.4110	135.6164	180.8219	226.0274	271.2329	316.4384	361.6438	406.8493	330
340	46.5753	93.1507	139.7260	186.3014	232.8767	279.4521	326.0274	372.6027	419.1781	340
350	47.9452	95.8904	143.8356	191.7808	239.7260	287.6712	335.6164	383.5616	431.5068	350
360	49.3151	98.6301	147.9452	197.2603	246.5753	295.8904	345.2055	394.5205	443.8356	360

TABLE III.—COMPOUND AMOUNT ON 1

$s = (1 + i)^n$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2 %	$2\frac{1}{2}\%$	3 %	n
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	1.025 1563	1.030 2250	1.040 4000	1.050 6250	1.060 9000	2
3	1.037 9707	1.045 6784	1.061 2080	1.076 8906	1.092 7270	3
4	1.050 9453	1.061 3636	1.082 4322	1.103 8129	1.125 5088	4
5	1.064 0822	1.077 2840	1.104 0808	1.131 4082	1.159 2741	5
6	1.077 3832	1.093 4433	1.126 1624	1.159 6934	1.194 0523	6
7	1.090 8505	1.109 8449	1.148 6857	1.188 6858	1.229 8739	7
8	1.104 4861	1.126 4926	1.171 6594	1.218 4029	1.266 7701	8
9	1.118 2922	1.143 3900	1.195 0926	1.248 8630	1.304 7732	9
10	1.132 2708	1.160 5408	1.218 9944	1.280 0845	1.343 9164	10
11	1.146 4242	1.177 9489	1.243 3743	1.312 0867	1.384 2339	11
12	1.160 7545	1.195 6182	1.268 2418	1.344 8888	1.425 7609	12
13	1.175 2639	1.213 5524	1.293 6066	1.378 5110	1.468 5337	13
14	1.189 9547	1.231 7557	1.319 4788	1.412 9738	1.512 5897	14
15	1.204 8292	1.250 2321	1.345 8683	1.448 2982	1.557 9674	15
16	1.219 8895	1.268 9855	1.372 7857	1.484 5056	1.604 7064	16
17	1.235 1382	1.288 0203	1.400 2414	1.521 6183	1.652 8476	17
18	1.250 5774	1.307 3406	1.428 2463	1.559 6587	1.702 4331	18
19	1.266 2096	1.326 9507	1.456 8112	1.598 6502	1.753 5061	19
20	1.282 0372	1.346 8550	1.485 9474	1.638 6164	1.806 1112	20
21	1.298 0627	1.367 0578	1.515 6663	1.679 5819	1.860 2946	21
22	1.314 2885	1.387 5637	1.545 9797	1.721 5714	1.916 1034	22
23	1.330 7171	1.408 3771	1.576 8993	1.764 6107	1.973 5865	23
24	1.347 3511	1.429 5028	1.608 4373	1.808 7260	2.032 7941	24
25	1.364 1929	1.450 9454	1.640 6060	1.853 9441	2.093 7780	25
26	1.381 2454	1.472 7095	1.673 4181	1.900 2927	2.156 5913	26
27	1.398 5109	1.494 8002	1.706 8865	1.947 8000	2.221 2890	27
28	1.415 9923	1.517 2222	1.741 0242	1.996 4950	2.287 9277	28
29	1.433 6922	1.539 9805	1.775 8447	2.046 4074	2.356 5655	29
30	1.451 6134	1.563 0802	1.811 3616	2.097 5676	2.427 2625	30
31	1.469 7585	1.586 5264	1.847 5888	2.150 0068	2.500 0803	31
32	1.488 1305	1.610 3243	1.884 5406	2.203 7569	2.575 0828	32
33	1.506 7321	1.634 4792	1.922 2314	2.258 8509	2.652 3352	33
34	1.525 5663	1.658 9964	1.960 6760	2.315 3221	2.731 9053	34
35	1.544 6359	1.683 8813	1.999 8896	2.373 2052	2.813 8625	35
36	1.563 9438	1.709 1395	2.039 8873	2.432 5353	2.898 2783	36
37	1.583 4931	1.734 7766	2.080 6851	2.493 3487	2.985 2267	37
38	1.603 2868	1.760 7983	2.122 2988	2.555 6824	3.074 7835	38
39	1.623 3279	1.787 2102	2.164 7448	2.619 5745	3.167 0270	39
40	1.643 6195	1.814 0184	2.208 0397	2.685 0638	3.262 0378	40
41	1.664 1647	1.841 2287	2.252 2005	2.752 1904	3.359 8989	41
42	1.684 9668	1.868 8471	2.297 2445	2.820 9952	3.460 6959	42
43	1.706 0289	1.896 8798	2.343 1894	2.891 5201	3.564 5168	43
44	1.727 3542	1.925 3330	2.390 0531	2.963 8081	3.671 4523	44
45	1.748 9461	1.954 2130	2.437 8542	3.037 9033	3.781 5958	45
46	1.770 8080	1.983 5262	2.486 6113	3.113 8509	3.895 0437	46
47	1.792 9431	2.013 2791	2.536 3435	3.191 6971	4.011 8950	47
48	1.815 3549	2.043 4783	2.587 0704	3.271 4896	4.132 2519	48
49	1.838 0468	2.074 1305	2.638 8118	3.353 2768	4.256 2194	49
50	1.861 0224	2.105 2424	2.691 5880	3.437 1087	4.383 9060	50

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	1.884 2851	2.136 8211	2.745 4198	3.523 0364	4.515 4232	**51**
52	1.907 8387	2.168 8734	2.800 3282	3.611 1123	4.650 8859	**52**
53	1.931 6867	2.201 4065	2.856 3347	3.701 3902	4.790 4125	**53**
54	1.955 8328	2.234 4276	2.913 4614	3.793 9249	4.934 1248	**54**
55	1.980 2807	2.267 9440	2.971 7307	3.888 7730	5.082 1486	**55**
56	2.005 0342	2.301 9631	3.031 1653	3.985 9924	5.234 6130	**56**
57	2.030 0971	2.336 4926	3.091 7886	4.085 6422	5.391 6514	**57**
58	2.055 4733	2.371 5400	3.156 6244	4.187 7832	5.553 4010	**58**
59	2.081 1668	2.407 1131	3.216 6969	4.292 4778	5.720 0030	**59**
60	2.107 1884	2.443 2198	3.281 0308	4.399 7897	5.891 6031	**60**
61	2.133 5211	2.479 8681	3.346 6514	4.509 7845	6.068 3512	**61**
62	2.160 1901	2.517 0661	3.413 5844	4.622 5291	6.250 4017	**62**
63	2.187 1925	2.554 8221	3.481 8561	4.738 0923	6.437 9138	**63**
64	2.214 5324	2.593 1444	3.551 4932	4.856 5446	6.631 0512	**64**
65	2.242 2141	2.632 0416	3.622 5231	4.977 9583	6.829 9827	**65**
66	2.270 2417	2.671 5222	3.694 9736	5.102 4072	7.034 8822	**66**
67	2.298 6198	2.711 5950	3.768 8730	5.229 9674	7.245 9287	**67**
68	2.327 3525	2.752 2690	3.844 2505	5.360 7166	7.463 3065	**68**
69	2.356 4444	2.793 5530	3.921 1355	5.494 7345	7.687 2057	**69**
70	2.385 9000	2.835 4563	3.999 5582	5.632 1029	7.917 8219	**70**
71	2.415 7237	2.877 9881	4.079 5494	5.772 9054	8.155 3566	**71**
72	2.445 9203	2.921 1580	4.161 1404	5.917 2281	8.400 0173	**72**
73	2.476 4943	2.964 9753	4.244 3632	6.065 1588	8.652 0178	**73**
74	2.507 4505	3.009 4500	4.329 2504	6.216 7877	8.911 5783	**74**
75	2.538 7936	3.054 5917	4.415 8355	6.372 2074	9.178 9257	**75**
76	2.570 5285	3.100 4106	4.504 1522	6.531 5126	9.454 2934	**76**
77	2.602 6601	3.146 9167	4.594 2352	6.694 8004	9.737 9222	**77**
78	2.635 1934	3.194 1205	4.686 1199	6.862 1704	10.030 0599	**78**
79	2.668 1333	3.242 0323	4.779 8423	7.033 7247	10.330 9617	**79**
80	2.701 4849	3.290 6628	4.875 4392	7.209 5678	10.640 8906	**80**
81	2.735 2535	3.340 0227	4.972 9479	7.389 8070	10.960 1173	**81**
82	2.769 4442	3.390 1231	5.072 4069	7.574 5522	11.288 9208	**82**
83	2.804 0622	3.440 9749	5.173 8550	7.763 9160	11.627 5884	**83**
84	2.839 1130	3.492 5895	5.277 3321	7.958 0139	11.976 4161	**84**
85	2.874 6019	3.544 9784	5.382 8788	8.156 9642	12.335 7085	**85**
86	2.910 5344	3.598 1531	5.490 5364	8.360 8883	12.705 7798	**86**
87	2.946 9161	3.652 1253	5.600 3471	8.569 9106	13.086 9532	**87**
88	2.983 7526	3.706 9072	5.712 3540	8.784 1583	13.479 5618	**88**
89	3.021 0495	3.762 5108	5.826 6011	9.003 7623	13.883 9487	**89**
90	3.058 8126	3.818 9485	5.943 1331	9.228 8563	14.300 4671	**90**
91	3.097 0477	3.876 2327	6.061 9958	9.459 5777	14.729 4811	**91**
92	3.135 7609	3.934 3762	6.183 2357	9.696 0672	15.171 3656	**92**
93	3.174 9579	3.993 3919	6.306 9004	9.938 4689	15.626 5065	**93**
94	3.214 6448	4.053 2927	6.433 0384	10.186 9306	16.095 3017	**94**
95	3.254 8279	4.114 0921	6.561 6992	10.441 6038	16.578 1608	**95**
96	3.295 5132	4.175 8035	6.692 9332	10.702 6439	17.075 5056	**96**
97	3.336 7072	4.238 4406	6.826 7918	10.970 2100	17.587 7708	**97**
98	3.378 4160	4.302 0172	6.963 3277	11.244 4653	18.115 4039	**98**
99	3.420 6462	4.366 5474	7.102 5942	11.525 5769	18.658 8660	**99**
100	3.463 4043	4.432 0457	7.244 6461	11.813 7164	19.218 6320	**100**

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	1
2	1.071 2250	1.081 6000	1.092 0250	1.102 5000	1.123 6000	2
3	1.108 7179	1.124 8640	1.141 1661	1.157 6250	1.191 0160	3
4	1.147 5230	1.169 8586	1.192 5186	1.215 5063	1.262 4770	4
5	1.187 6863	1.216 6529	1.246 1819	1.276 2816	1.338 2256	5
6	1.229 2553	1.265 3190	1.302 2601	1.340 0956	1.418 5191	6
7	1.272 2793	1.315 9318	1.360 8618	1.407 1004	1.503 6303	7
8	1.316 8090	1.368 5691	1.422 1006	1.477 4554	1.593 8481	8
9	1.362 8974	1.423 3118	1.486 0951	1.551 3282	1.689 4790	9
10	1.410 5988	1.480 2443	1.552 9694	1.628 8946	1.790 8477	10
11	1.459 9697	1.539 4541	1.622 8530	1.710 3394	1.898 2986	11
12	1.511 0687	1.601 0322	1.695 8814	1.795 8563	2.012 1965	12
13	1.563 9561	1.665 0735	1.772 1961	1.885 6491	2.132 9283	13
14	1.618 6945	1.731 6764	1.851 9449	1.979 9316	2.260 9040	14
15	1.675 3488	1.800 9435	1.935 2824	2.078 9282	2.396 5582	15
16	1.733 9860	1.872 9812	2.022 3702	2.182 8746	2.540 3517	16
17	1.794 6756	1.947 9005	2.113 3768	2.292 0183	2.692 7728	17
18	1.857 4892	2.025 8165	2.208 4788	2.406 6192	2.854 3392	18
19	1.922 5013	2.106 8492	2.307 8603	2.526 9502	3.025 5995	19
20	1.989 7889	2.191 1231	2.411 7140	2.653 2977	3.207 1355	20
21	2.059 4315	2.278 7681	2.520 2412	2.785 9626	3.399 5636	21
22	2.131 5116	2.369 9188	2.633 6520	2.925 2607	3.603 5374	22
23	2.206 1145	2.464 7155	2.752 1663	3.071 5238	3.819 7497	23
24	2.283 3285	2.563 3042	2.876 0138	3.225 0999	4.048 9346	24
25	2.363 2450	2.665 8363	3.005 4345	3.386 3549	4.291 8707	25
26	2.445 9586	2.772 4698	3.140 6790	3.555 6727	4.549 3830	26
27	2.531 5671	2.883 3686	3.282 0096	3.733 4563	4.822 3459	27
28	2.620 1720	2.998 7033	3.429 7000	3.920 1291	5.111 6867	28
29	2.711 8780	3.118 6515	3.584 0365	4.116 1356	5.418 3879	29
30	2.806 7937	3.243 3975	3.745 3181	4.321 9424	5.743 4912	30
31	2.905 0315	3.373 1334	3.913 8575	4.538 0395	6.088 1006	31
32	3.006 7076	3.508 0587	4.089 9810	4.764 9415	6.453 3867	32
33	3.111 9424	3.648 3811	4.274 0302	5.003 1885	6.840 5899	33
34	3.220 8603	3.794 3163	4.466 3615	5.253 3480	7.251 0253	34
35	3.333 5904	3.946 0890	4.667 3478	5.516 0154	7.686 0868	35
36	3.450 2661	4.103 9326	4.877 3785	5.791 8161	8.147 2520	36
37	3.571 0254	4.268 0899	5.096 8605	6.081 4069	8.636 0871	37
38	3.696 0113	4.438 8135	5.326 2192	6.385 4773	9.154 2523	38
39	3.825 3717	4.616 3660	5.565 8991	6.704 7512	9.703 5075	39
40	3.959 2597	4.801 0206	5.816 3645	7.039 9887	10.285 7179	40
41	4.097 8338	4.993 0615	6.078 1009	7.391 9881	10.902 8610	41
42	4.241 2580	5.192 7839	6.351 6155	7.761 5876	11.557 0327	42
43	4.389 7020	5.400 4953	6.637 4382	8.149 6669	12.250 4546	43
44	4.543 3416	5.616 5151	6.936 1229	8.557 1503	12.985 4819	44
45	4.702 3586	5.841 1757	7.248 2484	8.985 0078	13.764 6108	45
46	4.866 9411	6.074 8227	7.574 4196	9.434 2582	14.590 4875	46
47	5.037 2840	6.317 8156	7.915 2685	9.905 9711	15.465 9167	47
48	5.213 5890	6.570 5282	8.271 4556	10.401 2696	16.393 8717	48
49	5.396 0646	6.833 3494	8.643 6711	10.921 3331	17.377 5040	49
50	5.584 9269	7.106 6833	9.032 6363	11.467 3998	18.420 1543	50

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	5.780 3993	7.390 9507	9.439 1049	12.040 7698	19.525 3635	**51**
52	5.982 7133	7.686 5887	9.863 8646	12.642 8083	20.696 8853	**52**
53	6.192 1082	7.994 0523	10.307 7385	13.274 9487	21.938 6985	**53**
54	6.408 8320	8.313 8143	10.771 5868	13.938 6961	23.255 0204	**54**
55	6.633 1411	8.646 3669	11.256 3082	14.635 6309	24.650 3216	**55**
56	6.865 3011	8.992 2216	11.762 8420	15.367 4125	26.129 3409	**56**
57	7.105 5866	9.351 9105	12.292 1699	16.135 7831	27.697 1013	**57**
58	7.354 2822	9.725 9869	12.845 3176	16.942 5722	29.358 9274	**58**
59	7.611 6820	10.115 0264	13.423 3569	17.789 7009	31.120 4631	**59**
60	7.878 0909	10.519 6274	14.027 4079	18.~~697 1859~~ .679 1859	32.987 6909	**60**
61	8.153 8241	10.940 4125	14.658 6413	19.613 1452	34.966 9523	**61**
62	8.439 2079	11.378 0290	15.318 2801	20.593 8025	37.064 9694	**62**
63	8.734 5802	11.833 1502	16.007 6028	21.623 4926	39.288 8676	**63**
64	9.040 2905	12.306 4762	16.727 9449	22.704 6672	41.646 1997	**64**
65	9.356 7007	12.798 7352	17.480 7024	23.839 9006	44.144 9717	**65**
66	9.684 1852	13.310 6846	18.267 3340	25.031 8956	46.793 6699	**66**
67	10.023 1317	13.843 1120	19.089 3640	26.283 4904	49.601 2901	**67**
68	10.373 9413	14.396 8365	19.948 3854	27.597 6649	52.577 3676	**68**
69	10.737 0292	14.972 7100	20.846 0628	28.977 5481	55.732 0096	**69**
70	11.112 8253	15.571 6184	21.784 1356	30.426 4255	59.075 9302	**70**
71	11.501 7741	16.194 4831	22.764 4217	31.947 7468	62.620 4860	**71**
72	11.904 3362	16.842 2624	23.788 8207	33.545 1342	66.377 7152	**72**
73	12.320 9880	17.515 9529	24.859 3176	35.222 3909	70.360 3781	**73**
74	12.752 2226	18.216 5910	25.977 9869	36.983 5104	74.582 0007	**74**
75	13.198 5504	18.945 2547	27.146 9963	38.832 6859	79.056 9208	**75**
76	13.660 4996	19.703 0648	28.368 6111	40.774 3202	83.800 3360	**76**
77	14.138 6171	20.491 1874	29.645 1986	42.813 0362	88.828 3562	**77**
78	14.633 4687	21.310 8349	30.979 2326	44.953 6880	94.158 0576	**78**
79	15.145 6401	22.163 2683	32.373 2980	47.201 3724	99.807 5410	**79**
80	15.675 7375	23.049 7991	33.830 0964	49.561 4411	105.795 9935	**80**
81	16.224 3884	23.971 7910	35.352 4508	52.039 5131	112.143 7531	**81**
82	16.792 2419	24.930 6627	36.943 3111	54.641 4888	118.872 3782	**82**
83	17.379 9704	25.927 8892	38.605 7601	57.373 5632	126.004 7210	**83**
84	17.988 2694	26.965 0047	40.343 0193	60.242 2414	133.565 0042	**84**
85	18.617 8588	28.043 6049	42.158 4551	63.254 3534	141.578 9045	**85**
86	19.269 4839	29.165 3491	44.055 5856	66.417 0711	150.073 6388	**86**
87	19.943 9158	30.331 9631	46.038 0870	69.737 9247	159.078 0571	**87**
88	20.641 9529	31.545 2416	48.109 8009	73.224 8209	168.622 7405	**88**
89	21.364 4212	32.807 0513	50.274 7419	76.886 0620	178.740 1049	**89**
90	22.112 1759	34.119 3333	52.537 1053	80.730 3651	189.464 5112	**90**
91	22.886 1021	35.484 1067	54.901 2750	84.766 8833	200.832 3819	**91**
92	23.687 1157	36.903 4709	57.371 8324	89.005 2275	212.882 3248	**92**
93	24.516 1647	38.379 6098	59.953 5649	93.455 4888	225.655 2643	**93**
94	25.374 2305	39.914 7942	62.651 4753	98.128 2633	239.194 5801	**94**
95	26.262 3286	41.511 3859	65.470 7917	103.034 6765	253.546 2550	**95**
96	27.181 5101	43.171 8414	68.416 9773	108.186 4103	268.759 0303	**96**
97	28.132 8629	44.898 7150	71.495 7413	113.595 7308	284.884 5721	**97**
98	29.117 5131	46.694 6636	74.713 0496	119.275 5173	301.977 6464	**98**
99	30.136 6261	48.562 4502	78.075 1369	125.239 2932	320.096 3052	**99**
100	31.191 4080	50.504 9482	81.588 5180	131.501 2579	339.302 0835	**100**

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1+i)^{-n}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	**1**
2	0.975 4611	0.970 6617	0.961 1688	0.951 8144	0.942 5959	**2**
3	0.963 4183	0.956 3170	0.942 3223	0.928 5994	0.915 1417	**3**
4	0.951 5243	0.942 1842	0.923 8454	0.905 9506	0.888 4870	**4**
5	0.939 7771	0.928 2603	0.905 7308	0.883 8543	0.862 6088	**5**
6	0.928 1749	0.914 5422	0.887 9714	0.862 2969	0.837 4843	**6**
7	0.916 7159	0.901 0268	0.870 5602	0.841 2652	0.813 0915	**7**
8	0.905 3985	0.887 7111	0.853 4904	0.820 7466	0.789 4092	**8**
9	0.894 2207	0.874 5922	0.836 7553	0.800 7284	0.766 4167	**9**
10	0.883 1809	0.861 6672	0.820 3483	0.781 1984	0.744 0939	**10**
11	0.872 2775	0.848 9332	0.804 2630	0.762 1448	0.722 4213	**11**
12	0.861 5086	0.836 3874	0.788 4932	0.743 5559	0.701 3799	**12**
13	0.850 8727	0.824 0270	0.773 0325	0.725 4204	0.680 9513	**13**
14	0.840 3681	0.811 8493	0.757 8750	0.707 7272	0.661 1178	**14**
15	0.829 9932	0.799 8515	0.743 0147	0.690 4656	0.641 8619	**15**
16	0.819 7463	0.788 0310	0.728 4458	0.673 6249	0.623 1669	**16**
17	0.809 6260	0.776 3853	0.714 1626	0.657 1951	0.605 0164	**17**
18	0.799 6306	0.764 9116	0.700 1594	0.641 1659	0.587 3946	**18**
19	0.789 7587	0.756 6075	0.686 4308	0.625 5277	0.570 2860	**19**
20	0.780 0085	0.742 4704	0.672 9713	0.610 2709	0.553 6758	**20**
21	0.770 3788	0.731 4980	0.659 7758	0.595 3863	0.537 5493	**21**
22	0.760 8680	0.720 6876	0.646 8390	0.580 8647	0.521 8925	**22**
23	0.751 4745	0.710 0371	0.634 1559	0.566 6972	0.506 6917	**23**
24	0.742 1971	0.699 5439	0.621 7215	0.552 8754	0.491 9337	**24**
25	0.733 0341	0.689 2058	0.609 5309	0.539 3906	0.477 6056	**25**
26	0.723 9843	0.679 0205	0.597 5793	0.526 2347	0.463 6947	**26**
27	0.715 0463	0.668 9857	0.585 8620	0.513 3997	0.450 1891	**27**
28	0.706 2185	0.659 0993	0.574 3746	0.500 8778	0.437 0768	**28**
29	0.697 4998	0.649 3589	0.563 1123	0.488 6613	0.424 3464	**29**
30	0.688 8887	0.639 7624	0.552 0709	0.476 7427	0.411 9868	**30**
31	0.680 3839	0.630 3078	0.541 2460	0.465 1148	0.399 9871	**31**
32	0.671 9841	0.620 9929	0.530 6333	0.453 7705	0.388 3370	**32**
33	0.663 6880	0.611 8157	0.520 2287	0.442 7030	0.377 0262	**33**
34	0.655 4943	0.602 7741	0.510 0282	0.431 9053	0.366 0449	**34**
35	0.647 4018	0.593 8661	0.500 0276	0.421 3711	0.355 3834	**35**
36	0.639 4092	0.585 0897	0.490 2232	0.411 0937	0.345 0324	**36**
37	0.631 5152	0.576 4431	0.480 6109	0.401 0670	0.334 9829	**37**
38	0.623 7187	0.567 9242	0.471 1872	0.391 2849	0.325 2262	**38**
39	0.616 0185	0.559 5313	0.461 9482	0.381 7414	0.315 7535	**39**
40	0.608 4133	0.551 2623	0.452 8904	0.372 4306	0.306 5568	**40**
41	0.600 9021	0.543 1156	0.444 0102	0.363 3469	0.297 6280	**41**
42	0.593 4835	0.535 0893	0.435 3041	0.354 4848	0.288 9592	**42**
43	0.586 1566	0.527 1815	0.426 7688	0.345 8389	0.280 5429	**43**
44	0.578 9201	0.519 3907	0.418 4007	0.337 4038	0.272 3718	**44**
45	0.571 7729	0.511 7149	0.410 1968	0.329 1744	0.264 4386	**45**
46	0.564 7140	0.504 1527	0.402 1537	0.321 1458	0.256 7365	**46**
47	0.557 7422	0.496 7021	0.394 2684	0.313 3129	0.249 2588	**47**
48	0.550 8565	0.489 3617	0.386 5376	0.305 6712	0.241 9988	**48**
49	0.544 0558	0.482 1298	0.378 9584	0.298 2158	0.234 9503	**49**
50	0.537 3391	0.475 0047	0.371 5279	0.290 9422	0.228 0171	**50**

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	0.530 7052	0.467 9849	0.364 2430	0.283 8461	0.221 4632	**51**
52	0.524 1533	0.461 0689	0.357 1010	0.276 9230	0.215 0128	**52**
53	0.517 6823	0.454 2550	0.350 0990	0.270 1688	0.208 7503	**53**
54	0.511 2912	0.447 5419	0.343 2343	0.263 5793	0.202 6702	**54**
55	0.504 9789	0.440 9280	0.336 5042	0.257 1505	0.196 7672	**55**
56	0.498 7446	0.434 4118	0.329 9061	0.250 8785	0.191 0361	**56**
57	0.492 5873	0.427 9919	0.323 4374	0.244 7596	0.185 4719	**57**
58	0.486 5059	0.421 6669	0.317 0955	0.238 7898	0.180 0698	**58**
59	0.480 4997	0.415 4354	0.310 8779	0.232 9657	0.174 8251	**59**
60	0.474 5676	0.409 2960	0.304 7823	0.227 2836	0.169 7331	**60**
61	0.468 7087	0.403 2473	0.298 8061	0.221 7401	0.164 7894	**61**
62	0.462 9222	0.397 2879	0.292 9472	0.216 3318	0.159 9897	**62**
63	0.457 2071	0.391 4167	0.287 2031	0.211 0554	0.155 3298	**63**
64	0.451 5626	0.385 6322	0.281 5717	0.205 9077	0.150 8057	**64**
65	0.445 9878	0.379 9332	0.276 0507	0.200 8856	0.146 4133	**65**
66	0.440 4817	0.374 3184	0.270 6379	0.195 9859	0.142 1488	**66**
67	0.435 0437	0.368 7866	0.265 3313	0.191 2058	0.138 0085	**67**
68	0.429 6728	0.363 3366	0.260 1287	0.186 5422	0.133 9889	**68**
69	0.424 3682	0.357 9671	0.255 0282	0.181 9924	0.130 0863	**69**
70	0.419 1291	0.352 6769	0.250 0276	0.177 5536	0.126 2974	**70**
71	0.413 9546	0.347 4650	0.245 1251	0.173 2230	0.122 6188	**71**
72	0.408 8441	0.342 3300	0.240 3187	0.168 9980	0.119 0474	**72**
73	0.403 7966	0.337 2709	0.235 6066	0.164 8761	0.115 5800	**73**
74	0.398 8115	0.332 2866	0.230 9869	0.160 8548	0.112 2136	**74**
75	0.393 8879	0.327 3760	0.226 4577	0.156 9315	0.108 9452	**75**
76	0.389 0251	0.322 5379	0.222 0174	0.153 1039	0.105 7721	**76**
77	0.384 2223	0.317 7714	0.217 6641	0.149 3697	0.102 6913	**77**
78	0.379 4788	0.313 0752	0.213 3962	0.145 7265	0.099 7003	**78**
79	0.374 7939	0.308 4485	0.209 2119	0.142 1722	0.096 7964	**79**
80	0.370 1668	0.303 8901	0.205 1097	0.138 7046	0.093 9771	**80**
81	0.365 5968	0.299 3992	0.201 0880	0.135 3215	0.091 2399	**81**
82	0.361 0833	0.294 9745	0.197 1451	0.132 0210	0.088 5824	**82**
83	0.356 6255	0.290 6153	0.193 2795	0.128 8010	0.086 0024	**83**
84	0.352 2227	0.286 3205	0.189 4897	0.125 6595	0.083 4974	**84**
85	0.347 8743	0.282 0892	0.185 7742	0.122 5946	0.081 0655	**85**
86	0.343 5795	0.277 9204	0.182 1316	0.119 6045	0.078 7043	**86**
87	0.339 3378	0.273 8132	0.178 5604	0.116 6873	0.076 4120	**87**
88	0.335 1484	0.269 7667	0.175 0592	0.113 8413	0.074 1864	**88**
89	0.331 0108	0.265 7800	0.171 6266	0.111 0647	0.072 0256	**89**
90	0.326 9242	0.261 8522	0.168 2614	0.108 3558	0.069 9278	**90**
91	0.322 8881	0.257 9824	0.164 9622	0.105 7130	0.067 8911	**91**
92	0.318 9019	0.254 1699	0.161 7276	0.103 1346	0.065 9136	**92**
93	0.314 9648	0.250 4137	0.158 5565	0.100 6191	0.063 9938	**93**
94	0.311 0764	0.246 7130	0.155 4475	0.098 1650	0.062 1299	**94**
95	0.307 2359	0.243 0670	0.152 3995	0.095 7707	0.060 3203	**95**
96	0.303 4429	0.239 4749	0.149 4113	0.093 4349	0.058 5634	**96**
97	0.299 6967	0.235 9358	0.146 4817	0.091 1560	0.056 8577	**97**
98	0.295 9967	0.232 4491	0.143 6095	0.088 9326	0.055 2016	**98**
99	0.292 3424	0.229 0139	0.140 7936	0.086 7635	0.053 5938	**99**
100	0.288 7333	0.225 6294	0.138 0330	0.084 6474	0.052 0328	**100**

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1+i)^{-n}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	1
2	0.933 5107	0.924 5562	0.915 7300	0.907 0295	0.889 9964	2
3	0.901 9427	0.888 9964	0.876 2966	0.863 8376	0.839 6193	3
4	0.871 4422	0.854 8042	0.838 5613	0.822 7025	0.792 0937	4
5	0.841 9732	0.821 9271	0.802 4510	0.783 5262	0.747 2582	5
6	0.813 5006	0.790 3145	0.767 8957	0.746 2154	0.704 9605	6
7	0.785 9910	0.759 9178	0.734 8285	0.710 6813	0.665 0571	7
8	0.759 4116	0.730 6902	0.703 1851	0.676 8394	0.627 4124	8
9	0.733 7310	0.702 5867	0.672 9044	0.644 6089	0.591 8985	9
10	0.708 9188	0.675 5642	0.643 9277	0.613 9133	0.558 3948	10
11	0.684 9457	0.649 5809	0.616 1987	0.584 6793	0.526 7875	11
12	0.661 7833	0.624 5970	0.589 6639	0.556 8374	0.496 9694	12
13	0.639 4042	0.600 5741	0.564 2716	0.530 3214	0.468 8390	13
14	0.617 7818	0.577 4751	0.539 9729	0.505 0680	0.442 3010	14
15	0.596 8906	0.555 2645	0.516 7204	0.481 0171	0.417 2651	15
16	0.576 7059	0.533 9082	0.494 4693	0.458 1115	0.393 6463	16
17	0.557 2038	0.513 3732	0.473 1764	0.436 2967	0.371 3644	17
18	0.538 3611	0.493 6281	0.452 8004	0.415 5207	0.350 3438	18
19	0.520 1557	0.474 6424	0.433 3018	0.395 7340	0.330 5130	19
20	0.502 5659	0.456 3869	0.414 6429	0.376 8895	0.311 8047	20
21	0.485 5709	0.438 8336	0.396 7874	0.358 9424	0.294 1554	21
22	0.469 1506	0.421 9554	0.379 7009	0.341 8499	0.277 5051	22
23	0.453 2856	0.405 7263	0.363 3501	0.325 5713	0.261 7973	23
24	0.437 9571	0.390 1215	0.347 7035	0.310 0679	0.246 9785	24
25	0.423 1470	0.375 1168	0.332 7306	0.295 3028	0.232 9986	25
26	0.408 8377	0.360 6892	0.318 4025	0.281 2407	0.219 8100	26
27	0.395 0122	0.346 8166	0.304 6914	0.267 8483	0.207 3679	27
28	0.381 6543	0.333 4775	0.291 5707	0.255 0936	0.195 6301	28
29	0.368 7482	0.320 6514	0.279 0150	0.242 9463	0.184 5567	29
30	0.356 2784	0.308 3187	0.267 0000	0.231 3774	0.174 1101	30
31	0.344 2303	0.296 4603	0.255 5024	0.220 3595	0.164 2548	31
32	0.332 5897	0.285 0579	0.244 4999	0.209 8662	0.154 9574	32
33	0.321 3427	0.274 0942	0.233 9712	0.199 8725	0.146 1862	33
34	0.310 4761	0.263 5521	0.223 8959	0.190 3548	0.137 9115	34
35	0.299 9769	0.253 4155	0.214 2544	0.181 2903	0.130 1052	35
36	0.289 8327	0.243 6687	0.205 0282	0.172 6574	0.122 7408	36
37	0.280 0316	0.234 2968	0.196 1992	0.164 4356	0.115 7932	37
38	0.270 5619	0.225 2854	0.187 7504	0.156 6054	0.109 2389	38
39	0.261 4125	0.216 6206	0.179 6655	0.149 1480	0.103 0555	39
40	0.252 5725	0.208 2890	0.171 9287	0.142 0457	0.097 2222	40
41	0.244 0314	0.200 2779	0.164 5251	0.135 2816	0.091 7190	41
42	0.235 7791	0.192 5749	0.157 4403	0.128 8396	0.086 5274	42
43	0.227 8059	0.185 1682	0.150 6605	0.122 7044	0.081 6296	43
44	0.220 1023	0.178 0463	0.144 1728	0.116 8613	0.077 0091	44
45	0.212 6592	0.171 1984	0.137 9644	0.111 2965	0.072 6501	45
46	0.205 4679	0.164 6139	0.132 0233	0.105 9967	0.068 5378	46
47	0.198 5197	0.158 2826	0.126 3381	0.100 9492	0.064 6583	47
48	0.191 8065	0.152 1948	0.120 8977	0.096 1421	0.060 9984	48
49	0.185 3202	0.146 3411	0.115 6916	0.091 5639	0.057 5457	49
50	0.179 0534	0.140 7126	0.110 7096	0.087 2037	0.054 2884	50

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1+i)^{-n}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	0.172 9984	0.135 3006	0.105 9422	0.083 0512	0.051 2154	**51**
52	0.167 1482	0.130 0967	0.101 3801	0.079 0964	0.048 3164	**52**
53	0.161 4959	0.125 0930	0.097 0145	0.075 3299	0.045 5816	**53**
54	0.156 0347	0.120 2817	0.092 8368	0.071 7427	0.043 0015	**54**
55	0.150 7581	0.115 6555	0.088 8391	0.068 3264	0.040 5674	**55**
56	0.145 6600	0.111 2072	0.085 0135	0.065 0728	0.038 2712	**56**
57	0.140 7343	0.106 9300	0.081 3526	0.061 9741	0.036 1049	**57**
58	0.135 9752	0.102 8173	0.077 8494	0.059 0229	0.034 0612	**58**
59	0.131 3770	0.098 8628	0.074 4970	0.056 2123	0.032 1332	**59**
60	0.126 9343	0.095 0604	0.071 2890	0.053 5355	0.030 3143	**60**
61	0.122 6418	0.091 4042	0.068 2191	0.050 9862	0.028 5984	**61**
62	0.118 4945	0.087 8887	0.065 2815	0.048 5583	0.026 9797	**62**
63	0.114 4875	0.084 5084	0.062 4703	0.046 2460	0.025 4525	**63**
64	0.110 6159	0.081 2580	0.059 7802	0.044 0438	0.024 0118	**64**
65	0.106 8753	0.078 1327	0.057 2059	0.041 9465	0.022 6526	**65**
66	0.103 2611	0.075 1276	0.054 7425	0.039 9490	0.021 3704	**66**
67	0.099 7692	0.072 2381	0.052 3852	0.038 0467	0.020 1608	**67**
68	0.096 3954	0.069 4597	0.050 1294	0.036 2349	0.019 0196	**68**
69	0.093 1356	0.066 7882	0.047 9707	0.034 5095	0.017 9430	**69**
70	0.089 9861	0.064 2194	0.045 9050	0.032 8662	0.016 9274	**70**
71	0.086 9431	0.061 7494	0.043 9282	0.031 3011	0.015 9692	**71**
72	0.084 0030	0.059 3744	0.042 0366	0.029 8106	0.015 0653	**72**
73	0.081 1623	0.057 0908	0.040 2264	0.028 3910	0.014 2125	**73**
74	0.078 4177	0.054 8950	0.038 4941	0.027 0391	0.013 4081	**74**
75	0.075 7659	0.052 7837	0.036 8365	0.025 7515	0.012 6491	**75**
76	0.073 2038	0.050 7535	0.035 2502	0.024 5252	0.011 9331	**76**
77	0.070 7283	0.048 8015	0.033 7323	0.023 3574	0.011 2577	**77**
78	0.068 3365	0.046 9245	0.032 2797	0.022 2451	0.010 6204	**78**
79	0.066 0256	0.045 1197	0.030 8897	0.021 1858	0.010 0193	**79**
80	0.063 7929	0.043 3843	0.029 5595	0.020 1770	0.009 4522	**80**
81	0.061 6356	0.041 7157	0.028 2866	0.019 2162	0.008 9171	**81**
82	0.059 5513	0.040 1112	0.027 0685	0.018 3011	0.008 4124	**82**
83	0.057 5375	0.038 5685	0.025 9029	0.017 4296	0.007 9362	**83**
84	0.055 5918	0.037 0851	0.024 7874	0.016 5996	0.007 4870	**84**
85	0.053 7119	0.035 6588	0.023 7200	0.015 8092	0.007 0632	**85**
86	0.051 8955	0.034 2873	0.022 6986	0.015 0564	0.006 6634	**86**
87	0.050 1406	0.032 9685	0.021 7211	0.014 3394	0.006 2862	**87**
88	0.048 4450	0.031 7005	0.020 7858	0.013 6566	0.005 9304	**88**
89	0.046 8068	0.030 4813	0.019 8907	0.013 0063	0.005 5947	**89**
90	0.045 2240	0.029 3089	0.019 0342	0.012 3869	0.005 2780	**90**
91	0.043 6946	0.028 1816	0.018 2145	0.011 7971	0.004 9793	**91**
92	0.042 2170	0.027 0977	0.017 4302	0.011 2353	0.004 6974	**92**
93	0.040 7894	0.026 0555	0.016 6796	0.010 7003	0.004 4315	**93**
94	0.039 4101	0.025 0534	0.015 9613	0.010 1907	0.004 1807	**94**
95	0.038 0774	0.024 0898	0.015 2740	0.009 7055	0.003 9441	**95**
96	0.036 7897	0.023 1632	0.014 6163	0.009 2433	0.003 7208	**96**
97	0.035 5456	0.022 2724	0.013 9868	0.008 8031	0.003 5102	**97**
98	0.034 3436	0.021 4157	0.013 3845	0.008 3840	0.003 3115	**98**
99	0.033 1822	0.020 5920	0.012 8082	0.007 9847	0.003 1241	**99**
100	0.032 0601	0.019 8000	0.012 2566	0.007 6045	0.002 9473	**100**

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	**1**
2	2.012 5000	2.015 0000	2.020 0000	2.025 0000	2.030 0000	**2**
3	3.037 6562	3.045 2250	3.060 4000	3.075 6250	3.090 9000	**3**
4	4.075 6269	4.090 9034	4.121 6080	4.152 5156	4.183 6270	**4**
5	5.126 5723	5.152 2669	5.204 0402	5.256 3285	5.309 1358	**5**
6	6.190 6544	6.229 5509	6.308 1210	6.387 7367	6.468 4099	**6**
7	7.268 0376	7.322 9942	7.434 2834	7.547 4301	7.662 4622	**7**
8	8.358 8881	8.432 8391	8.582 9691	8.736 1159	8.892 3360	**8**
9	9.463 3742	9.559 3317	9.754 6284	9.954 5188	10.159 1061	**9**
10	10.581 6664	10.702 7217	10.949 7210	11.203 3818	11.463 8793	**10**
11	11.713 9372	11.863 2625	12.168 7154	12.483 4663	12.807 7957	**11**
12	12.860 3614	13.041 2114	13.412 0897	13.795 5530	14.192 0296	**12**
13	14.021 1159	14.236 8296	14.680 3315	15.140 4418	15.617 7904	**13**
14	15.196 3799	15.450 3820	15.973 9382	16.518 9528	17.086 3242	**14**
15	16.386 3346	16.682 1378	17.293 4169	17.931 9267	18.598 9139	**15**
16	17.591 1638	17.932 3698	18.639 2853	19.380 2248	20.156 8813	**16**
17	18.811 0534	19.201 3554	20.012 0710	20.864 7304	21.761 5877	**17**
18	20.046 1915	20.489 3757	21.412 3124	22.386 3487	23.414 4354	**18**
19	21.296 7689	21.796 7164	22.840 5586	23.946 0074	25.116 8684	**19**
20	22.562 9785	23.123 6671	24.297 3698	25.544 6576	26.870 3745	**20**
21	23.845 0158	24.470 5221	25.783 3172	27.183 2741	28.676 4857	**21**
22	25.143 0785	25.837 5799	27.298 9835	28.862 8559	30.536 7803	**22**
23	26.457 3669	27.225 1436	28.844 9632	30.584 4273	32.452 8837	**23**
24	27.788 0840	28.633 5208	30.421 8625	32.349 0380	34.426 4702	**24**
25	29.135 4351	30.063 0236	32.030 2997	34.157 7639	36.459 2643	**25**
26	30.499 6280	31.513 9690	33.670 9057	36.011 7080	38.553 0423	**26**
27	31.880 8734	32.986 6785	35.344 3238	37.912 0007	40.709 6335	**27**
28	33.279 3843	34.481 4787	37.051 2103	39.859 8007	42.930 9225	**28**
29	34.695 3766	35.998 7009	38.792 2345	41.856 2958	45.218 8502	**29**
30	36.129 0688	37.538 6814	40.568 0792	43.902 7032	47.575 4157	**30**
31	37.580 6822	39.101 7616	42.379 4408	46.000 2707	50.002 6782	**31**
32	39.050 4407	40.688 2880	44.227 0296	48.150 2775	52.502 7585	**32**
33	40.538 5712	42.298 6123	46.111 5702	50.354 0344	55.077 8413	**33**
34	42.045 3033	43.933 0915	48.033 8016	52.612 8853	57.730 1765	**34**
35	43.570 8696	45.592 0879	49.994 4776	54.928 2074	60.462 0818	**35**
36	45.115 5055	47.275 9692	51.994 3672	57.301 4126	63.275 9443	**36**
37	46.679 4493	48.985 1087	54.034 2545	59.733 9479	66.174 2226	**37**
38	48.262 9424	50.719 8854	56.114 9396	62.227 2966	69.159 4493	**38**
39	49.866 2292	52.480 6837	58.237 2384	64.782 9791	72.234 2328	**39**
40	51.489 5571	54.267 8939	60.401 9832	67.402 5535	75.401 2597	**40**
41	53.133 1765	56.081 9123	62.610 0228	70.087 6174	78.663 2975	**41**
42	54.797 3412	57.923 1410	64.862 2233	72.839 8078	82.023 1965	**42**
43	56.482 3080	59.791 9881	67.159 4678	75.660 8030	85.483 8923	**43**
44	58.188 3369	61.688 8679	69.502 6571	78.552 3231	89.048 4091	**44**
45	59.915 6911	63.614 2010	71.892 7103	81.516 1312	92.719 8614	**45**
46	61.664 6372	65.568 4140	74.330 5645	84.554 0344	96.501 4572	**46**
47	63.435 4452	67.551 9402	76.817 1758	87.667 8853	100.396 5009	**47**
48	65.228 3884	69.565 2193	79.353 5193	90.859 5824	104.408 3960	**48**
49	67.043 7431	71.608 6976	81.940 5897	94.131 0720	108.540 6479	**49**
50	68.881 7899	73.682 8280	84.579 4015	97.484 3288	112.796 8673	**50**

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	70.742 8123	75.788 0705	87.270 9895	100.921 4575	117.180 7733	51
52	72.627 0974	77.924 8915	90.016 4093	104.444 4939	121.696 1965	52
53	74.534 9361	80.093 7649	92.816 7375	108.055 6063	126.347 0824	53
54	76.466 6228	82.295 1714	95.673 0722	111.756 9965	131.137 4949	54
55	78.422 4556	84.529 5989	98.586 5337	115.550 9214	136.071 6197	55
56	80.402 7363	86.797 5429	101.558 2643	119.439 6944	141.153 7683	56
57	82.407 7705	89.099 5061	104.589 4296	123.425 6868	146.388 3814	57
58	84.437 8676	91.435 9987	107.681 2182	127.511 3289	151.780 0328	58
59	86.493 3410	93.807 5386	110.834 8426	131.699 1121	157.333 4338	59
60	88.574 5078	96.214 6517	114.051 5394	135.991 5900	163.053 4368	60
61	90.681 6891	98.657 8715	117.332 5702	140.391 3797	168.945 0399	61
62	92.815 2102	101.137 7396	120.679 2216	144.901 1642	175.013 3911	62
63	94.975 4003	103.654 8057	124.092 8060	149.523 6933	181.263 7928	63
64	97.162 5928	106.209 6277	127.574 6622	154.261 7856	187.701 7066	64
65	99.377 1253	108.802 7722	131.126 1554	159.118 3303	194.332 7578	65
66	101.619 3393	111.434 8137	134.748 6785	164.096 2885	201.162 7406	66
67	103.889 5811	114.106 3359	138.443 6521	169.198 6957	208.197 6228	67
68	106.188 2008	116.817 9310	142.212 5251	174.428 6631	215.443 5515	68
69	108.515 5533	119.570 1999	146.056 7756	179.789 3797	222.906 8580	69
70	110.871 9978	122.363 7529	149.977 9111	185.284 1142	230.594 0637	70
71	113.257 8977	125.199 2092	153.977 4694	190.916 2171	238.511 8856	71
72	115.673 6215	128.077 1974	158.057 0188	196.689 1225	246.667 2422	72
73	118.119 5417	130.998 3553	162.218 1591	202.606 3506	255.067 2595	73
74	120.596 0360	133.963 3307	166.462 5223	208.671 5093	263.719 2773	74
75	123.103 4864	136.972 7806	170.791 7728	214.888 2970	272.630 8556	75
76	125.642 2800	140.027 3723	175.207 6082	221.260 5045	281.809 7813	76
77	128.212 8085	143.127 7829	179.711 7604	227.792 0171	291.264 0747	77
78	130.815 4686	146.274 6997	184.305 9956	234.486 8175	301.001 9969	78
79	133.450 6620	149.468 8202	188.992 1155	241.348 9879	311.032 0568	79
80	136.118 7953	152.710 8525	193.771 9578	248.382 7126	321.363 0185	80
81	138.820 2802	156.001 5153	198.647 3970	255.592 2805	332.003 9091	81
82	141.555 5337	159.341 5380	203.620 3449	262.982 0875	342.964 0264	82
83	144.324 9779	162.731 6611	208.692 7518	270.556 6397	354.252 9472	83
84	147.129 0401	166.172 6360	213.866 6068	278.320 5557	365.880 5356	84
85	149.968 1531	169.665 2255	219.143 9390	286.278 5695	377.856 9517	85
86	152.842 7550	173.210 2039	224.526 8177	294.435 5338	390.192 6602	86
87	155.753 2895	176.808 3569	230.017 3541	302.796 4221	402.898 4400	87
88	158.700 2056	180.460 4823	235.617 7012	311.366 3327	415.985 3932	88
89	161.683 9581	184.167 3895	241.330 0552	320.150 4910	429.464 9550	89
90	164.705 0076	187.929 9004	247.156 6563	329.154 2533	443.348 9037	90
91	167.763 8202	191.748 8489	253.099 7894	338.383 1096	457.649 3708	91
92	170.860 8680	195.625 0816	259.161 7852	347.842 6873	472.378 8519	92
93	173.996 6288	199.559 4578	265.345 0209	357.538 7545	487.550 2174	93
94	177.171 5867	203.552 8497	271.651 9214	367.477 2234	503.176 7240	94
95	180.386 2315	207.606 1425	278.084 9598	377.664 1540	519.272 0257	95
96	183.641 0594	211.720 2346	284.646 6590	388.105 7578	535.850 1865	96
97	186.936 5726	215.896 0381	291.339 5922	398.808 4018	552.925 6920	97
98	190.273 2798	220.134 4787	298.166 3840	409.778 6118	570.513 4628	98
99	193.651 6958	224.436 4959	305.129 7117	421.023 0771	588.628 8667	99
100	197.072 3420	228.803 0433	312.232 3059	432.548 6540	607.287 7327	100

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	3½%	4%	4½%	5%	6%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.035 0000	2.040 0000	2.045 0000	2.050 0000	2.060 0000	2
3	3.106 2250	3.121 6000	3.137 0250	3.152 5000	3.183 6000	3
4	4.214 9429	4.246 4640	4.278 1911	4.310 1250	4.374 6160	4
5	5.362 4659	5.416 3226	5.470 7097	5.525 6312	5.637 0930	5
6	6.550 1522	6.632 9755	6.716 8917	6.801 9128	6.975 3185	6
7	7.779 4075	7.898 2945	8.019 1518	8.142 0085	8.393 8376	7
8	9.051 6868	9.214 2263	9.380 0136	9.549 1089	9.897 4679	8
9	10.368 4958	10.582 7953	10.802 1142	11.026 5643	11.491 3160	9
10	11.731 3932	12.006 1071	12.288 2094	12.577 8925	13.180 7949	10
11	13.141 9919	13.486 3514	13.841 1788	14.206 7872	14.971 6426	11
12	14.601 9616	15.025 8055	15.464 0318	15.917 1265	16.869 9412	12
13	16.113 0303	16.626 8377	17.159 9132	17.712 9828	18.882 1377	13
14	17.676 9864	18.291 9112	18.932 0194	19.598 6320	21.015 0659	14
15	19.295 6809	20.023 5876	20.784 0543	21.578 5636	23.275 9699	15
16	20.971 0297	21.824 5311	22.719 3367	23.657 4918	25.672 5281	16
17	22.705 0157	23.697 5124	24.741 7069	25.840 3664	28.212 8798	17
18	24.499 6913	25.645 4129	26.855 0837	28.132 3847	30.905 6525	18
19	26.357 1805	27.671 2294	29.063 5625	30.539 0039	33.759 9917	19
20	28.279 6818	29.778 0786	31.371 4228	33.065 9541	36.785 5912	20
21	30.269 4707	31.969 2017	33.783 1368	35.719 2518	39.992 7267	21
22	32.328 9021	34.247 9698	36.303 3780	38.505 2144	43.392 2903	22
23	34.460 4137	36.617 8886	38.937 0300	41.430 4751	46.995 8277	23
24	36.666 5282	39.082 6041	41.689 1963	44.501 9989	50.815 5774	24
25	38.949 8567	41.645 9083	44.565 2101	47.727 0988	54.864 5120	25
26	41.313 1017	44.311 7446	47.570 6446	51.113 4538	59.156 3827	26
27	43.759 0602	47.084 2144	50.711 3236	54.669 1264	63.705 7657	27
28	46.290 6273	49.967 5830	53.993 3332	58.402 5828	68.528 1116	28
29	48.910 7993	52.966 2863	57.423 0332	62.322 7119	73.639 7983	29
30	51.622 6773	56.084 9377	61.007 0697	66.438 8475	79.058 1862	30
31	54.429 4710	59.328 3353	64.752 3878	70.760 7899	84.801 6774	31
32	57.334 5025	62.701 4687	68.666 2452	75.298 8294	90.889 7780	32
33	60.341 2101	66.209 5274	72.756 2263	80.063 7708	97.343 1647	33
34	63.453 1524	69.857 9085	77.030 2565	85.066 9594	104.183 7546	34
35	66.674 0127	73.652 2249	81.496 6180	90.320 3074	111.434 7799	35
36	70.007 6032	77.598 3138	86.163 9658	95.836 3227	119.120 8667	36
37	73.457 8693	81.702 2464	91.041 3443	101.628 1389	127.268 1187	37
38	77.028 8947	85.970 3363	96.138 2048	107.709 5458	135.904 2058	38
39	80.724 9060	90.409 1497	101.464 4240	114.095 0231	145.058 4581	39
40	84.550 2777	95.025 5157	107.030 3231	120.799 7742	154.761 9656	40
41	88.509 5375	99.826 5363	112.846 6876	127.839 7630	165.047 6836	41
42	92.607 3713	104.819 5978	118.924 7885	135.231 7511	175.950 5446	42
43	96.848 6293	110.012 3817	125.276 4040	142.993 3387	187.507 5772	43
44	101.238 3313	115.412 8770	131.913 8422	151.143 0056	199.758 0319	44
45	105.781 6729	121.029 3920	138.849 9651	159.700 1559	212.743 5138	45
46	110.484 0314	126.870 5677	146.098 2135	168.685 1637	226.508 1246	46
47	115.350 9725	132.945 3904	153.672 6331	178.119 4218	241.098 6121	47
48	120.388 2566	139.263 2060	161.587 9016	188.025 3929	256.564 5288	48
49	125.601 8456	145.833 7343	169.859 3572	198.426 6626	272.958 4005	49
50	130.997 9102	152.667 0837	178.503 0283	209.347 9957	290.335 9046	50

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	136.582 8370	159.773 7670	187.535 6646	220.815 3955	308.756 0589	**51**
52	142.363 2363	167.164 7177	196.974 7695	232.856 1653	328.281 4224	**52**
53	148.345 9496	174.851 3064	206.838 6341	245.498 9735	348.978 3077	**53**
54	154.538 0578	182.845 3586	217.146 3726	258.773 9222	370.917 0062	**54**
55	160.946 8898	191.159 1730	227.917 9594	272.712 6183	394.172 0266	**55**
56	166.580 0310	199.805 5399	239.174 2676	287.348 2492	418.882 3482	**56**
57	174.445 3321	208.797 7615	250.937 1096	302.715 6617	444.951 6890	**57**
58	181.550 9187	218.149 6720	263.229 2795	318.851 4448	472.648 7904	**58**
59	188.905 2008	227.875 6588	276.074 5971	335.794 0170	502.007 7178	**59**
60	196.516 8829	237.990 6852	289.497 9540	353.583 7179	533.128 1809	**60**
61	204.394 9738	248.510 3126	303.525 3619	372.262 9038	566.115 8717	**61**
62	212.548 7979	259.450 7251	318.184 0032	391.876 0490	601.082 8240	**62**
63	220.988 0058	270.828 7541	333.502 2833	412.469 8514	638.147 7935	**63**
64	229.722 5860	282.661 9043	349.509 8861	434.093 3440	677.436 6611	**64**
65	238.762 8765	294.968 3805	366.237 8310	456.798 0112	719.082 8608	**65**
66	248.119 5772	307.767 1157	383.718 5333	480.637 9117	763.227 8324	**66**
67	257.803 7624	321.077 8003	401.985 8673	505.669 8073	810.021 5024	**67**
68	267.826 8941	334.920 9123	421.075 2314	531.953 2977	859.622 7925	**68**
69	278.200 8354	349.317 7488	441.023 6168	559.550 9626	912.200 1600	**69**
70	288.937 8646	364.290 4588	461.869 6795	588.528 5107	967.932 1696	**70**
71	300.050 6899	379.862 0771	483.653 8151	618.954 9362	1027.008 0998	**71**
72	311.552 4640	396.056 5602	506.418 2368	650.902 6831	1089.628 5858	**72**
73	323.456 8002	412.898 8226	530.207 0575	684.447 8172	1156.006 3010	**73**
74	335.777 7882	430.414 7755	555.066 3751	719.670 2081	1226.366 6790	**74**
75	348.530 0108	448.631 3665	581.044 3619	756.653 7185	1300.948 6798	**75**
76	361.728 5612	467.576 6212	608.191 3582	795.486 4044	1380.005 6006	**76**
77	375.389 0609	487.279 6860	636.559 9694	836.260 7246	1463.805 9366	**77**
78	389.527 6780	507.770 8735	666.205 1680	879.073 7608	1552.634 2928	**78**
79	404.161 1467	529.081 7084	697.184 4005	924.027 4489	1646.792 3503	**79**
80	419.306 7868	551.244 9767	729.557 6985	971.228 8213	1746.599 8914	**80**
81	434.982 5244	574.294 7758	763.387 7950	1020.790 2624	1852.395 8849	**81**
82	451.206 9127	598.266 5668	798.740 2457	1072.829 7755	1964.539 6379	**82**
83	467.999 1547	623.197 2295	835.683 5568	1127.471 2643	2083.412 0162	**83**
84	485.379 1251	649.125 1187	874.289 3169	1184.844 8275	2209.416 7372	**84**
85	503.367 3945	676.090 1235	914.632 3361	1245.087 0689	2342.981 7414	**85**
86	521.985 2533	704.133 7284	956.790 7912	1308.341 4223	2484.560 6459	**86**
87	541.254 7372	733.299 0775	1000.846 3769	1374.758 4935	2634.634 2847	**87**
88	561.198 6530	763.631 0406	1046.884 4638	1444.496 4181	2793.712 3417	**88**
89	581.840 6058	795.176 2823	1094.994 2647	1517.721 2390	2962.335 0822	**89**
90	603.205 0270	827.983 3335	1145.269 0066	1594.607 3010	3141.075 1872	**90**
91	625.317 2030	862.102 6669	1197.806 1119	1675.337 6660	3330.539 6984	**91**
92	648.203 3051	897.586 7736	1252.707 3869	1760.104 5493	3531.372 0803	**92**
93	671.890 4207	934.490 2445	1310.079 2193	1849.109 7768	3744.254 4051	**93**
94	696.406 5855	972.869 8543	1370.032 7842	1942.565 2656	3969.909 6694	**94**
95	721.780 8160	1012.784 6485	1432.684 2595	2040.693 5289	4209.104 2496	**95**
96	748.043 1445	1054.296 0344	1498.155 0512	2143.728 2054	4462.650 5046	**96**
97	775.224 6546	1097.467 8758	1566.572 0285	2251.914 6156	4731.409 5348	**97**
98	803.357 5175	1142.366 5908	1638.067 7698	2365.510 3464	5016.294 1070	**98**
99	832.475 0306	1189.061 2544	1712.780 8194	2484.785 8637	5318.271 7534	**99**
100	862.611 6567	1237.623 7046	1790.855 9563	2610.025 1569	5638.368 0586	**100**

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n|}} = \frac{1 - v^n}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	1
2	1.963 1154	1.955 8834	1.941 5609	1.927 4242	1.913 4697	2
3	2.926 5307	2.912 2004	2.883 8833	2.856 0236	2.828 6114	3
4	3.878 0580	3.854 3846	3.807 7287	3.761 9742	3.717 0984	4
5	4.817 8350	4.782 6450	4.713 4595	4.645 8285	4.579 7072	5
6	5.746 0099	5.697 1872	5.601 4309	5.508 1254	5.417 1914	6
7	6.662 7258	6.598 2140	6.471 9911	6.349 3906	6.230 2830	7
8	7.568 1243	7.485 9251	7.325 4814	7.170 1372	7.019 6922	8
9	8.462 3450	8.360 5173	8.162 2367	7.970 8655	7.786 1089	9
10	9.345 5259	9.222 1846	8.982 5850	8.752 0639	8.530 2028	10
11	10.217 8034	10.071 1178	9.786 8480	9.514 2087	9.252 6241	11
12	11.079 3120	10.907 5052	10.575 3412	10.257 7646	9.954 0040	12
13	11.930 1847	11.731 5322	11.348 3737	10.983 1850	10.634 9553	13
14	12.770 5527	12.543 3815	12.106 2488	11.690 9122	11.296 0731	14
15	13.600 5459	13.343 2330	12.849 2635	12.381 3777	11.937 9351	15
16	14.420 2923	14.131 2641	13.577 7093	13.055 0027	12.561 1020	16
17	15.229 9183	14.907 6493	14.291 8719	13.712 1977	13.166 1185	17
18	16.029 5489	15.672 5609	14.992 0312	14.353 3636	13.753 5131	18
19	16.819 3076	16.426 1684	15.678 4620	14.978 8913	14.323 7991	19
20	17.599 3161	17.168 6388	16.351 4333	15.589 1623	14.877 4749	20
21	18.369 6949	17.900 1367	17.011 2092	16.184 5486	15.415 0241	21
22	19.130 5629	18.620 8244	17.658 0482	16.765 4132	15.936 9166	22
23	19.882 0374	19.330 8614	18.292 2041	17.332 1105	16.443 6084	23
24	20.624 2345	20.030 4054	18.913 9256	17.884 9858	16.935 5421	24
25	21.357 2686	20.719 6112	19.523 4565	18.424 3764	17.413 1477	25
26	22.081 2530	21.398 6317	20.121 0358	18.950 6111	17.876 8424	26
27	22.796 2993	22.067 6175	20.706 8978	19.464 0109	18.327 0315	27
28	23.502 5178	22.726 7167	21.281 2724	19.964 8887	18.764 1082	28
29	24.200 0176	23.376 0756	21.844 3847	20.453 5499	19.188 4546	29
30	24.888 9062	24.015 8380	22.396 4556	20.930 2926	19.600 4413	30
31	25.569 2901	24.646 1458	22.937 7015	21.395 4074	20.000 4285	31
32	26.241 2742	25.267 1387	23.468 3348	21.849 1780	20.388 7655	32
33	26.904 9621	25.878 9544	23.988 5636	22.291 8809	20.765 7918	33
34	27.560 4564	26.481 7285	24.498 5917	22.723 7863	21.131 8367	34
35	28.207 8582	27.075 5946	24.998 6193	23.145 1573	21.487 2201	35
36	28.847 2674	27.660 6843	25.488 8425	23.556 2511	21.832 2525	36
37	29.478 7826	28.237 1274	25.969 4534	23.957 3181	22.167 2354	37
38	30.102 5013	28.805 0516	26.440 6406	24.348 6030	22.492 4616	38
39	30.718 5198	29.364 5829	26.902 5888	24.730 3444	22.808 2151	39
40	31.326 9332	29.915 8452	27.355 4792	25.102 7750	23.114 7720	40
41	31.927 8352	30.458 9608	27.799 4895	25.466 1220	23.412 4000	41
42	32.521 3187	30.994 0500	28.234 7936	25.820 6068	23.701 3592	42
43	33.107 4753	31.521 2316	28.661 5623	26.166 4457	23.981 9021	43
44	33.686 3954	32.040 6222	29.079 9631	26.503 8495	24.254 2739	44
45	34.258 1682	32.552 3372	29.490 1599	26.833 0239	24.518 7125	45
46	34.822 8822	33.056 4898	29.892 3136	27.154 1696	24.775 4491	46
47	35.380 6244	33.553 1919	30.286 5820	27.467 4826	25.024 7078	47
48	35.931 4809	34.042 5536	30.673 1196	27.773 1537	25.266 7066	48
49	36.475 5367	34.524 6834	31.052 0780	28.071 3695	25.501 6569	49
50	37.012 8757	34.999 6881	31.423 6059	28.362 3117	25.729 7640	50

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	37.543 5810	35.467 6730	31.787 8489	28.646 1577	25.951 2272	**51**
52	38.067 7343	35.928 7418	32.144 9499	28.923 0807	26.166 2400	**52**
53	38.585 4166	36.382 9969	32.495 0489	29.193 2495	26.374 9903	**53**
54	39.096 7078	36.830 5388	32.838 2833	29.456 8288	26.577 6605	**54**
55	39.601 6867	37.271 4668	33.174 7875	29.713 9793	26.774 4276	**55**
56	40.100 4313	37.705 8786	33.504 6936	29.964 8578	26.965 4637	**56**
57	40.593 0185	38.133 8706	33.828 1310	30.209 6174	27.150 9357	**57**
58	41.079 5245	38.555 5375	34.145 2265	30.448 4072	27.331 0055	**58**
59	41.560 0242	38.970 9729	34.456 1044	30.681 3729	27.505 8306	**59**
60	42.034 5918	39.380 2689	34.760 8867	30.908 6565	27.675 5637	**60**
61	42.503 3005	39.783 5161	35.059 6928	31.130 3966	27.840 3531	**61**
62	42.966 2227	40.180 8041	35.352 6400	31.346 7284	28.000 3428	**62**
63	43.423 4299	40.572 2208	35.639 8432	31.557 7838	28.155 6726	**63**
64	43.874 9925	40.957 8530	35.921 4149	31.763 6915	28.306 4783	**64**
65	44.320 9802	41.337 7862	36.197 4655	31.964 5771	28.452 8915	**65**
66	44.761 4619	41.712 1046	36.468 1035	32.160 5630	28.595 0403	**66**
67	45.196 5056	42.080 8912	36.733 4348	32.351 7688	28.733 0488	**67**
68	45.626 1784	42.444 2278	36.993 5635	32.538 3110	28.867 0377	**68**
69	46.050 5466	42.802 1949	37.248 5917	32.720 3034	28.997 1240	**69**
70	46.469 6756	43.154 8718	37.498 6193	32.897 8570	29.123 4214	**70**
71	46.883 6302	43.502 3368	37.743 7444	33.071 0800	29.246 0401	**71**
72	47.292 4743	43.844 6668	37.984 0631	33.240 0780	29.365 0875	**72**
73	47.696 2709	44.181 9577	38.219 6697	33.404 9542	29.480 6675	**73**
74	48.095 0824	44.514 2243	38.450 6566	33.565 8089	29.592 8811	**74**
75	48.488 9703	44.841 6003	38.677 1143	33.722 7404	29.701 8263	**75**
76	48.877 9953	45.164 1383	38.899 1317	33.875 8443	29.807 5983	**76**
77	49.262 2176	45.481 9096	39.116 7958	34.025 2140	29.910 2896	**77**
78	49.641 6964	45.794 9848	39.330 1919	34.170 9405	30.009 9899	**78**
79	50.016 4903	46.103 4333	39.539 4039	34.313 1127	30.106 7863	**79**
80	50.386 6571	46.407 3235	39.744 5136	34.451 8172	30.200 7634	**80**
81	50.752 2539	46.706 7226	39.945 6016	34.587 1388	30.292 0033	**81**
82	51.113 3372	47.001 6972	40.142 7466	34.719 1598	30.380 5858	**82**
83	51.469 9626	47.292 3125	40.336 0261	34.847 9607	30.466 5881	**83**
84	51.822 1853	47.578 6330	40.525 5158	34.973 6202	30.550 0856	**84**
85	52.170 0596	47.860 7222	40.711 2900	35.096 2149	30.631 1510	**85**
86	52.513 6391	48.138 6425	40.893 4216	35.215 8194	30.709 8554	**86**
87	52.852 9769	48.412 4557	41.071 9819	35.332 5067	30.786 2673	**87**
88	53.188 1253	48.682 2224	41.247 0411	35.446 3480	30.860 4537	**88**
89	53.519 1361	48.948 0023	41.418 6677	35.557 4127	30.932 4794	**89**
90	53.846 0603	49.209 8545	41.586 9292	35.665 7685	31.002 4071	**90**
91	54.168 9485	49.467 8370	41.751 8913	35.771 4814	31.070 2982	**91**
92	54.487 8504	49.722 0069	41.913 6190	35.874 6160	31.136 2118	**92**
93	54.802 8152	49.972 4205	42.072 1754	35.975 2352	31.200 2057	**93**
94	55.113 8915	50.219 1335	42.227 6230	36.073 4002	31.262 3356	**94**
95	55.421 1274	50.462 2005	42.380 0225	36.169 1709	31.322 6559	**95**
96	55.724 5703	50.701 6754	42.529 4339	36.262 6057	31.381 2193	**96**
97	56.024 2670	50.937 6112	42.675 9155	36.353 7617	31.438 0770	**97**
98	56.320 2637	51.170 0603	42.819 5250	36.442 6943	31.493 2787	**98**
99	56.612 6061	51.399 0742	42.960 3187	36.529 4579	31.546 8725	**99**
100	56.901 3394	51.624 7037	43.098 3516	36.614 1053	31.598 9053	**100**

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	**1**
2	1.899 6943	1.886 0947	1.872 6678	1.859 4104	1.833 3927	**2**
3	2.801 6370	2.775 0910	2.748 9644	2.723 2480	2.673 0119	**3**
4	3.673 0792	3.629 8952	3.587 5257	3.545 9505	3.465 1056	**4**
5	4.515 0524	4.451 8223	4.389 9767	4.329 4767	4.212 3638	**5**
6	5.328 5530	5.242 1369	5.157 8725	5.075 6921	4.917 3243	**6**
7	6.114 5440	6.002 0547	5.892 7009	5.786 3734	5.582 3814	**7**
8	6.873 9555	6.732 7449	6.595 8861	6.463 2128	6.209 7938	**8**
9	7.607 6865	7.435 3316	7.268 7905	7.107 8217	6.801 6923	**9**
10	8.316 6053	8.110 8958	7.912 7182	7.721 7349	7.360 0871	**10**
11	9.001 5510	8.760 4767	8.528 9169	8.306 4142	7.886 8746	**11**
12	9.663 3343	9.385 0738	9.118 5808	8.863 2516	8.383 8439	**12**
13	10.302 7385	9.985 6478	9.682 8524	9.393 5730	8.852 6830	**13**
14	10.920 5203	10.563 1229	10.222 8253	9.898 6409	9.294 9839	**14**
15	11.517 4109	11.118 3874	10.739 5457	10.379 6580	9.712 2490	**15**
16	12.094 1168	11.652 2956	11.234 0150	10.837 7696	10.105 8953	**16**
17	12.651 3206	12.165 6689	11.707 1914	11.274 0662	10.477 2597	**17**
18	13.189 6817	12.659 2970	12.159 9918	11.689 5869	10.827 6035	**18**
19	13.709 8374	13.133 9394	12.593 2936	12.085 3209	11.158 1165	**19**
20	14.212 4033	13.590 3263	13.007 9365	12.462 2103	11.469 9212	**20**
21	14.697 9742	14.029 1599	13.404 7239	12.821 1527	11.764 0766	**21**
22	15.167 1248	14.451 1153	13.784 4248	13.163 0026	12.041 5817	**22**
23	15.620 4105	14.856 8417	14.147 7749	13.488 5739	12.303 3790	**23**
24	16.058 3676	15.246 9631	14.495 4784	13.798 6418	12.550 3575	**24**
25	16.481 5146	15.622 0799	14.828 2090	14.093 9446	12.783 3562	**25**
26	16.890 3523	15.982 7692	15.146 6114	14.375 1853	13.003 1662	**26**
27	17.285 3645	16.329 5857	15.451 3028	14.643 0336	13.210 5341	**27**
28	17.667 0188	16.663 0632	15.742 8735	14.898 1273	13.406 1643	**28**
29	18.035 7670	16.983 7146	16.021 8885	15.141 0736	13.590 7210	**29**
30	18.392 0454	17.292 0333	16.288 8885	15.372 4510	13.764 8312	**30**
31	18.736 2758	17.588 4936	16.544 3910	15.592 8105	13.929 0860	**31**
32	19.068 8655	17.873 5515	16.788 8909	15.802 6767	14.084 0434	**32**
33	19.390 2082	18.147 6457	17.022 8621	16.002 5492	14.230 2296	**33**
34	19.700 6842	18.411 1978	17.246 7580	16.192 9040	14.368 1411	**34**
35	20.000 6611	18.664 6132	17.461 0124	16.374 1943	14.498 2464	**35**
36	20.290 4938	18.908 2820	17.666 0406	16.546 8517	14.620 9871	**36**
37	20.570 5254	19.142 5788	17.862 2398	16.711 2873	14.736 7803	**37**
38	20.841 0874	19.367 8642	18.049 9902	16.867 8927	14.846 0192	**38**
39	21.102 4999	19.584 4848	18.229 6557	17.017 0407	14.949 0747	**39**
40	21.355 0723	19.792 7739	18.401 5844	17.159 0864	15.046 2969	**40**
41	21.599 1037	19.993 0518	18.566 1095	17.294 3680	15.138 0159	**41**
42	21.834 8828	20.185 6267	18.723 5498	17.423 2076	15.224 5433	**42**
43	22.062 6887	20.370 7949	18.874 2103	17.545 9120	15.306 1729	**43**
44	22.282 7910	20.548 8413	19.018 3831	17.662 7733	15.383 1820	**44**
45	22.495 4503	20.720 0397	19.156 3474	17.774 0698	15.455 8321	**45**
46	22.700 9181	20.884 6536	19.288 3707	17.880 0665	15.524 3699	**46**
47	22.899 4378	21.042 9361	19.414 7088	17.981 0157	15.589 0282	**47**
48	23.091 2443	21.195 1309	19.535 6065	18.077 1578	15.650 0266	**48**
49	23.276 5645	21.341 4720	19.651 2981	18.168 7217	15.707 5723	**49**
50	23.455 6179	21.482 1846	19.762 0078	18.255 9255	15.761 8606	**50**

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	23.628 6163	21.617 4852	19.867 9500	18.338 9766	15.813 0761	**51**
52	23.795 7645	21.747 5819	19.969 3302	18.418 0730	15.861 3925	**52**
53	23.957 2604	21.872 6749	20.066 3447	18.493 4028	15.906 9741	**53**
54	24.113 2951	21.992 9567	20.159 1815	18.565 1456	15.949 9755	**54**
55	24.264 0532	22.108 6122	20.248 0206	18.633 4720	15.990 5430	**55**
56	24.409 7133	22.219 8194	20.333 0340	18.698 5447	16.028 8141	**56**
57	24.550 4776	22.326 7494	20.414 3866	18.760 5188	16.064 9190	**57**
58	24.686 4228	22.429 5668	20.492 2360	18.819 5417	16.098 9802	**58**
59	24.817 7998	22.528 4296	20.566 7330	18.875 7540	16.131 1134	**59**
60	24.944 7341	22.623 4900	20.638 0220	18.929 2895	16.161 4277	**60**
61	25.067 3760	22.714 8942	20.706 2412	18.980 2757	16.190 0261	**61**
62	25.185 8705	22.802 7829	20.771 5227	19.028 8340	16.217 0058	**62**
63	25.300 3580	22.887 2912	20.833 9930	19.075 0800	16.242 4583	**63**
64	25.410 9739	22.968 5493	20.893 7732	19.119 1238	16.266 4701	**64**
65	25.517 8492	23.046 6820	20.950 9791	19.161 0703	16.289 1227	**65**
66	25.621 1103	23.121 8096	21.005 7217	19.201 0194	16.310 4931	**66**
67	25.720 8795	23.194 0477	21.058 1068	19.239 0661	16.330 6539	**67**
68	25.817 2749	23.263 5074	21.108 2362	19.275 3010	16.349 6735	**68**
69	25.910 4105	23.330 2956	21.156 2069	19.309 8105	16.367 6165	**69**
70	26.000 3966	23.394 5150	21.202 1119	19.342 6766	16.384 5439	**70**
71	26.087 3398	23.456 2644	21.246 0401	19.373 9778	16.400 5131	**71**
72	26.171 3428	23.515 6388	21.288 0766	19.403 7883	16.415 5784	**72**
73	26.252 5051	23.572 7297	21.328 3030	19.432 1794	16.429 7909	**73**
74	26.330 9228	23.627 6247	21.366 7971	19.459 2185	16.443 1990	**74**
75	26.406 6887	23.680 4083	21.403 6336	19.484 9700	16.455 8481	**75**
76	26.479 8924	23.731 1619	21.438 8838	19.509 4952	16.467 7812	**76**
77	26.550 6207	23.779 9633	21.472 6161	19.532 8526	16.479 0389	**77**
78	26.618 9572	23.826 8878	21.504 8958	19.555 0977	16.489 6593	**78**
79	26.684 9828	23.872 0075	21.535 7854	19.576 2835	16.499 6786	**79**
80	26.748 7757	23.915 3918	21.565 3449	19.596 4605	16.509 1308	**80**
81	26.810 4113	23.957 1075	21.593 6315	19.615 6767	16.518 0479	**81**
82	26.869 9626	23.997 2188	21.620 7000	19.633 9778	16.526 4603	**82**
83	26.927 5001	24.035 7873	21.646 6029	19.651 4074	16.534 3965	**83**
84	26.983 0919	24.072 8724	21.671 3903	19.668 0070	16.541 8835	**84**
85	27.036 8037	24.108 5312	21.695 1103	19.683 8162	16.548 9467	**85**
86	27.088 6993	24.142 8184	21.717 8089	19.698 8726	16.555 6101	**86**
87	27.138 8399	24.175 7869	21.739 5301	19.713 2120	16.561 8963	**87**
88	27.187 2849	24.207 4874	21.760 3159	19.726 8686	16.567 8267	**88**
89	27.234 0917	24.237 9687	21.780 2066	19.739 8748	16.573 4214	**89**
90	27.279 3156	24.267 2776	21.799 2407	19.752 2617	16.578 6994	**90**
91	27.323 0103	24.295 4592	21.817 4553	19.764 0588	16.583 6787	**91**
92	27.365 2273	24.322 5569	21.834 8854	19.775 2941	16.588 3762	**92**
93	27.406 0167	24.348 6124	21.851 5650	19.785 9944	16.592 8077	**93**
94	27.445 4268	24.373 6658	21.867 5263	19.796 1851	16.596 9884	**94**
95	27.483 5042	24.397 7556	21.882 8003	19.805 8906	16.600 9324	**95**
96	27.520 2939	24.420 9188	21.897 4166	19.815 1339	16.604 6532	**96**
97	27.555 8395	24.443 1912	21.911 4034	19.823 9370	16.608 1634	**97**
98	27.590 1831	24.464 6069	21.924 7879	19.832 3210	16.611 4749	**98**
99	27.623 3653	24.485 1990	21.937 5961	19.840 3057	16.614 5990	**99**
100	27.655 4254	24.504 9990	21.949 8527	19.847 9102	16.617 5462	**100**

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_{\overline{n}|}} = \frac{i}{1 - v^n}. \quad \left(\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i\right)$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	0.509 3944	0.511 2779	0.515 0495	0.518 8272	0.522 6108	2
3	0.341 7012	0.343 3830	0.346 7547	0.350 1372	0.353 5304	3
4	0.257 8610	0.259 4448	0.262 6238	0.265 8179	0.269 0270	4
5	0.207 5621	0.209 0893	0.212 1584	0.215 2469	0.218 3546	5
6	0.174 0338	0.175 5252	0.178 5258	0.181 5500	0.184 5975	6
7	0.150 0887	0.151 5562	0.154 5120	0.157 4954	0.160 5064	7
8	0.132 1331	0.133 5840	0.136 5098	0.139 4673	0.142 4564	8
9	0.118 1705	0.119 6098	0.122 5154	0.125 4569	0.128 4339	9
10	0.107 0031	0.108 4342	0.111 3265	0.114 2588	0.117 2305	10
11	0.097 8684	0.099 2938	0.102 1779	0.105 1060	0.108 0774	11
12	0.090 2583	0.091 6800	0.094 5596	0.097 4871	0.100 4621	12
13	0.083 8210	0.085 2404	0.088 1184	0.091 0483	0.094 0295	13
14	0.078 3051	0.079 7233	0.082 6020	0.085 5365	0.088 5263	14
15	0.073 5265	0.074 9444	0.077 8255	0.080 7665	0.083 7666	15
16	0.069 3467	0.070 7651	0.073 6501	0.076 5990	0.079 6108	16
17	0.065 6602	0.067 0797	0.069 9698	0.072 9278	0.075 9525	17
18	0.062 3848	0.063 8058	0.066 7021	0.069 6701	0.072 7087	18
19	0.059 4555	0.060 8785	0.063 7818	0.066 7606	0.069 8139	19
20	0.056 8204	0.058 2457	0.061 1567	0.064 1471	0.067 2157	20
21	0.054 4375	0.055 8655	0.058 7847	0.061 7873	0.064 8718	21
22	0.052 2724	0.053 7033	0.056 6314	0.059 6466	0.062 7474	22
23	0.050 2967	0.051 7308	0.054 6681	0.057 6964	0.060 8139	23
24	0.048 4866	0.049 9241	0.052 8711	0.055 9128	0.059 0474	24
25	0.046 8225	0.048 2635	0.051 2204	0.054 2759	0.057 4279	25
26	0.045 2873	0.046 7320	0.049 6992	0.052 7687	0.055 9383	26
27	0.043 8668	0.045 3153	0.048 2931	0.051 3769	0.054 5642	27
28	0.042 5486	0.044 0011	0.046 9897	0.050 0879	0.053 2932	28
29	0.041 3223	0.042 7788	0.045 7784	0.048 8913	0.052 1147	29
30	0.040 1785	0.041 6392	0.044 6499	0.047 7776	0.051 0193	30
31	0.039 1094	0.040 5743	0.043 5963	0.046 7390	0.049 9989	31
32	0.038 1079	0.039 5771	0.042 6106	0.045 7683	0.049 0466	32
33	0.037 1679	0.038 6414	0.041 6865	0.044 8594	0.048 1561	33
34	0.036 2839	0.037 7619	0.040 8187	0.044 0067	0.047 3220	34
35	0.035 4511	0.036 9336	0.040 0022	0.043 2056	0.046 5393	35
36	0.034 6653	0.036 1524	0.039 2329	0.042 4516	0.045 8038	36
37	0.033 9227	0.035 4144	0.038 5068	0.041 7409	0.045 1116	37
38	0.033 2198	0.034 7161	0.037 8206	0.041 0701	0.044 4593	38
39	0.032 5537	0.034 0546	0.037 1711	0.040 4362	0.043 8439	39
40	0.031 9214	0.033 4271	0.036 5557	0.039 8362	0.043 2624	40
41	0.031 3206	0.032 8311	0.035 9719	0.039 2679	0.042 7124	41
42	0.030 7491	0.032 2643	0.035 4173	0.038 7288	0.042 1917	42
43	0.030 2047	0.031 7246	0.034 8899	0.038 2169	0.041 6981	43
44	0.029 6856	0.031 2104	0.034 3879	0.037 7304	0.041 2298	44
45	0.029 1901	0.030 7198	0.033 9096	0.037 2675	0.040 7852	45
46	0.028 7167	0.030 2512	0.033 4534	0.036 8268	0.040 3625	46
47	0.028 2641	0.029 8034	0.033 0179	0.036 4067	0.039 9605	47
48	0.027 8307	0.029 3750	0.032 6018	0.036 0060	0.039 5778	48
49	0.027 4156	0.028 9648	0.032 2040	0.035 6235	0.039 2131	49
50	0.027 0176	0.028 5717	0.031 8232	0.035 2581	0.038 8655	50

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_{\overline{n}|}} = \frac{i}{1 - v^n} \quad \left(\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i\right)$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	0.026 6357	0.028 1947	0.031 4586	0.034 9087	0.038 5338	**51**
52	0.026 2690	0.027 8329	0.031 1091	0.034 5745	0.038 2172	**52**
53	0.025 9165	0.027 4854	0.030 7739	0.034 2545	0.037 9147	**53**
54	0.025 5776	0.027 1514	0.030 4523	0.033 9480	0.037 6256	**54**
55	0.025 2514	0.026 8302	0.030 1434	0.033 6542	0.037 3491	**55**
56	0.024 9374	0.026 5211	0.029 8466	0.033 3724	0.037 0845	**56**
57	0.024 6348	0.026 2234	0.029 5612	0.033 1020	0.036 8311	**57**
58	0.024 3430	0.025 9366	0.029 2867	0.032 8424	0.036 5885	**58**
59	0.024 0616	0.025 6601	0.029 0224	0.032 5931	0.036 3559	**59**
60	0.023 7899	0.025 3934	0.028 7680	0.032 3534	0.036 1330	**60**
61	0.023 5276	0.025 1360	0.028 5228	0.032 1229	0.035 9191	**61**
62	0.023 2741	0.024 8875	0.028 2864	0.031 9013	0.035 7139	**62**
63	0.023 0290	0.024 6474	0.028 0585	0.031 6879	0.035 5168	**63**
64	0.022 7920	0.024 4153	0.027 8385	0.031 4825	0.035 3276	**64**
65	0.022 5627	0.024 1909	0.027 6262	0.031 2846	0.035 1458	**65**
66	0.022 3406	0.023 9739	0.027 4212	0.031 0940	0.034 9711	**66**
67	0.022 1256	0.023 7638	0.027 2232	0.030 9102	0.034 8031	**67**
68	0.021 9172	0.023 5603	0.027 0317	0.030 7330	0.034 6416	**68**
69	0.021 7153	0.023 3633	0.026 8467	0.030 5621	0.034 4862	**69**
70	0.021 5194	0.023 1724	0.026 6676	0.030 3971	0.034 3366	**70**
71	0.021 3294	0.022 9873	0.026 4945	0.030 2379	0.034 1927	**71**
72	0.021 1450	0.022 8078	0.026 3268	0.030 0842	0.034 0540	**72**
73	0.020 9660	0.022 6337	0.026 1645	0.029 9357	0.033 9205	**73**
74	0.020 7921	0.022 4647	0.026 0074	0.029 7922	0.033 7919	**74**
75	0.020 6232	0.022 3007	0.025 8551	0.029 6536	0.033 6680	**75**
76	0.020 4591	0.022 1415	0.025 7075	0.029 5196	0.033 5485	**76**
77	0.020 2995	0.021 9868	0.025 5645	0.029 3900	0.033 4333	**77**
78	0.020 1444	0.021 8365	0.025 4258	0.029 2646	0.033 3222	**78**
79	0.019 9934	0.021 6904	0.025 2912	0.029 1434	0.033 2151	**79**
80	0.019 8465	0.021 5483	0.025 1607	0.029 0260	0.033 1117	**80**
81	0.019 7036	0.021 4102	0.025 0340	0.028 9125	0.033 0120	**81**
82	0.019 5644	0.021 2758	0.024 9111	0.028 8025	0.032 9158	**82**
83	0.019 4288	0.021 1451	0.124 7917	0.028 6961	0.032 8228	**83**
84	0.019 2968	0.021 0178	0.024 6758	0.028 5930	0.032 7331	**84**
85	0.019 1681	0.020 8940	0.024 5632	0.028 4931	0.032 6465	**85**
86	0.019 0427	0.020 7733	0.024 4538	0.028 3963	0.032 5628	**86**
87	0.018 9204	0.020 6558	0.024 3475	0.028 3025	0.032 4820	**87**
88	0.018 8012	0.020 5414	0.024 2442	0.028 2116	0.032 4039	**88**
89	0.018 6849	0.020 4298	0.024 1437	0.028 1235	0.032 3285	**89**
90	0.018 5715	0.020 3211	0.024 0460	0.028 0381	0.032 2556	**90**
91	0.018 4608	0.020 2152	0.023 9510	0.027 9552	0.032 1851	**91**
92	0.018 3527	0.020 1118	0.023 8586	0.027 8749	0.032 1169	**92**
93	0.018 2472	0.020 0110	0.023 7687	0.027 7969	0.032 0511	**93**
94	0.018 1442	0.019 9127	0.023 6812	0.027 7213	0.031 9874	**94**
95	0.018 0437	0.019 8168	0.023 5960	0.027 6479	0.031 9258	**95**
96	0.017 9454	0.019 7232	0.023 5131	0.027 5766	0.031 8662	**96**
97	0.017 8494	0.019 6319	0.023 4324	0.027 5075	0.031 8086	**97**
98	0.017 7556	0.019 5427	0.023 3538	0.027 4403	0.031 7528	**98**
99	0.017 6639	0.019 4556	0.023 2773	0.027 3752	0.031 6989	**99**
100	0.017 5743	0.019 3706	0.023 2027	0.027 3119	0.031 6467	**100**

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_{\overline{n}|}} = \frac{i}{1-v}. \quad \left(\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i\right)$$

n	3½%	4%	4½%	5%	6%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	1
2	0.526 4005	0.530 1961	0.533 9976	0.537 8049	0.545 4369	2
3	0.356 9342	0.360 3485	0.363 7734	0.367 2086	0.374 1098	3
4	0.272 2511	0.275 4900	0.278 7436	0.282 0118	0.288 5915	4
5	0.221 4814	0.224 6271	0.227 7916	0.230 9748	0.237 3964	5
6	0.187 6682	0.190 7619	0.193 8784	0.197 0175	0.203 3626	6
7	0.163 5445	0.166 6096	0.169 7015	0.172 8198	0.179 1350	7
8	0.145 4766	0.148 5278	0.151 6097	0.154 7218	0.161 0359	8
9	0.131 4460	0.134 4930	0.137 5745	0.140 6901	0.147 0222	9
10	0.120 2414	0.123 2909	0.126 3788	0.129 5046	0.135 8680	10
11	0.111 0920	0.114 1490	0.117 2482	0.120 3889	0.126 7929	11
12	0.103 4839	0.106 5522	0.109 6662	0.112 8254	0.119 2770	12
13	0.097 0616	0.100 1437	0.103 2754	0.106 4558	0.112 9601	13
14	0.091 5707	0.094 6690	0.097 8203	0.101 0240	0.107 5849	14
15	0.086 8251	0.089 9411	0.093 1138	0.096 3423	0.102 9628	15
16	0.082 6848	0.085 8200	0.089 0154	0.092 2699	0.098 9521	16
17	0.079 0431	0.082 1985	0.085 4176	0.088 6991	0.095 4448	17
18	0.075 8168	0.078 9933	0.082 2369	0.085 5462	0.092 3565	18
19	0.072 9403	0.076 1386	0.079 4073	0.082 7450	0.089 6209	19
20	0.070 3611	0.073 5817	0.076 8761	0.080 2426	0.087 1846	20
21	0.068 0366	0.071 2801	0.074 6006	0.077 9961	0.085 0046	21
22	0.065 9321	0.069 1988	0.072 5456	0.075 9705	0.083 0456	22
23	0.064 0188	0.067 3091	0.070 6825	0.074 1368	0.081 2785	23
24	0.062 2728	0.065 5868	0.068 9870	0.072 4709	0.079 6790	24
25	0.060 6740	0.064 0120	0.067 4390	0.070 9525	0.078 2267	25
26	0.059 2054	0.062 5674	0.066 0214	0.069 5643	0.076 9043	26
27	0.057 8524	0.061 2385	0.064 7195	0.068 2919	0.075 6972	27
28	0.056 6026	0.060 0130	0.063 5208	0.067 1225	0.074 5926	28
29	0.055 4454	0.058 8799	0.062 4146	0.066 0455	0.073 5796	29
30	0.054 3713	0.057 8301	0.061 3915	0.065 0514	0.072 6489	30
31	0.053 3724	0.056 8554	0.060 4434	0.064 1321	0.071 7922	31
32	0.052 4415	0.055 9486	0.059 5632	0.063 2804	0.071 0023	32
33	0.051 5724	0.055 1036	0.058 7445	0.062 4900	0.070 2729	33
34	0.050 7597	0.054 3148	0.057 9819	0.061 7554	0.069 5984	34
35	0.049 9983	0.053 5773	0.057 2704	0.061 0717	0.068 9739	35
36	0.049 2842	0.052 8869	0.056 6058	0.060 4345	0.068 3948	36
37	0.048 6132	0.052 2396	0.055 9840	0.059 8398	0.067 8574	37
38	0.047 9821	0.051 6319	0.055 4017	0.059 2842	0.067 3581	38
39	0.047 3878	0.051 0608	0.054 8557	0.058 7646	0.066 8938	39
40	0.046 8273	0.050 5235	0.054 3431	0.058 2782	0.066 4615	40
41	0.046 2982	0.050 0174	0.053 8616	0.057 8223	0.066 0589	41
42	0.045 7983	0.049 5402	0.053 4087	0.057 3947	0.065 6834	42
43	0.045 3254	0.049 0899	0.052 9823	0.056 9933	0.065 3331	43
44	0.044 8777	0.048 6645	0.052 5807	0.056 6163	0.065 0061	44
45	0.044 4534	0.048 2625	0.052 2020	0.056 2617	0.064 7005	45
46	0.044 0511	0.047 8820	0.051 8447	0.055 9282	0.064 4149	46
47	0.043 6692	0.047 5219	0.051 5073	0.055 6142	0.064 1477	47
48	0.043 3065	0.047 1806	0.051 1886	0.055 3184	0.063 8977	48
49	0.042 9617	0.046 8571	0.050 8872	0.055 0396	0.063 6636	49
50	0.042 6337	0.046 5502	0.050 6021	0.054 7767	0.063 4443	50

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_{\overline{n|}}} = \frac{i}{1 - v^n}. \quad \left(\frac{1}{s_{\overline{n|}}} = \frac{1}{a_{\overline{n|}}} - i\right)$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	0.042 3216	0.046 2588	0.050 3323	0.054 5287	0.063 2388	**51**
52	0.042 0243	0.045 9821	0.050 0768	0.054 2945	0.063 0462	**52**
53	0.041 7410	0.045 7191	0.049 8347	0.054 0733	0.062 8655	**53**
54	0.041 4709	0.045 4691	0.049 6052	0.053 8644	0.062 6960	**54**
55	0.041 2132	0.045 2312	0.049 3875	0.053 6669	0.062 5370	**55**
56	0.040 9673	0.045 0049	0.049 1811	0.053 4801	0.062 3876	**56**
57	0.040 7325	0.044 7893	0.048 9851	0.053 3034	0.062 2474	**57**
58	0.040 5081	0.044 5840	0.048 7990	0.053 1363	0.062 1157	**58**
59	0.040 2937	0.044 3884	0.048 6222	0.052 9780	0.061 9920	**59**
60	0.140 0886	0.044 2018	0.048 4543	0.052 8282	0.061 8757	**60**
61	0.039 8925	0.044 0240	0.048 2946	0.052 6823	0.061 7664	**61**
62	0.039 7048	0.043 8543	0.048 1428	0.052 5518	0.061 6637	**62**
63	0.039 5251	0.043 6924	0.047 9985	0.052 4244	0.061 5670	**63**
64	0.039 3531	0.043 5378	0.047 8611	0.052 3037	0.061 4762	**64**
65	0.039 1883	0.043 3902	0.047 7305	0.052 1892	0.061 3907	**65**
66	0.039 0303	0.043 2492	0.047 6061	0.052 0806	0.061 3102	**66**
67	0.038 8789	0.043 1145	0.047 4876	0.051 9776	0.061 2345	**67**
68	0.038 7337	0.042 9858	0.047 3749	0.051 8799	0.061 1633	**68**
69	0.038 5945	0.042 8627	0.047 2675	0.051 7872	0.061 0963	**69**
70	0.038 4609	0.042 7451	0.047 1651	0.051 6992	0.061 0331	**70**
71	0.038 3328	0.042 6325	0.047 0676	0.051 6156	0.060 9737	**71**
72	0.038 2097	0.042 5249	0.046 9747	0.051 5363	0.060 9177	**72**
73	0.038 0916	0.042 4219	0.046 8861	0.051 4610	0.060 8651	**73**
74	0.037 9782	0.042 3233	0.046 8016	0.051 3895	0.060 8154	**74**
75	0.037 8692	0.042 2290	0.046 7210	0.051 3216	0.060 7687	**75**
76	0.037 7645	0.042 1387	0.046 6442	0.051 2571	0.060 7246	**76**
77	0.037 6639	0.042 0522	0.046 5709	0.051 1958	0.060 6831	**77**
78	0.037 5672	0.041 9694	0.046 5010	0.051 1376	0.060 6441	**78**
79	0.037 4743	0.041 8901	0.046 4343	0.051 0822	0.060 6072	**79**
80	0.037 3849	0.041 8141	0.046 3707	0.051 0296	0.060 5725	**80**
81	0.037 2989	0.041 7413	0.046 3099	0.050 9796	0.060 5398	**81**
82	0.037 2163	0.041 6715	0.046 2520	0.050 9321	0.060 5090	**82**
83	0.037 1368	0.041 6046	0.046 1966	0.050 8869	0.060 4800	**83**
84	0.037 0602	0.041 5405	0.046 1438	0.050 8440	0.060 4526	**84**
85	0.036 9866	0.041 4791	0.046 0933	0.050 8032	0.060 4268	**85**
86	0.036 9158	0.041 4202	0.046 0452	0.050 7643	0.060 4024	**86**
87	0.036 8476	0.041 3637	0.045 9992	0.050 7274	0.060 3796	**87**
88	0.036 7819	0.041 3095	0.045 9552	0.050 6923	0.060 3580	**88**
89	0.036 7187	0.041 2576	0.045 9132	0.050 6589	0.060 3376	**89**
90	0.036 6578	0.041 2078	0.045 8732	0.050 6271	0.060 3184	**90**
91	0.036 5992	0.041 1600	0.045 8349	0.050 5969	0.060 3003	**91**
92	0.036 5427	0.041 1141	0.045 7983	0.050 5681	0.060 2832	**92**
93	0.036 4883	0.041 0701	0.045 7633	0.050 5408	0.060 2671	**93**
94	0.036 4359	0.041 0279	0.045 7299	0.050 5148	0.060 2519	**94**
95	0.036 3855	0.040 9874	0.045 6980	0.050 4900	0.060 2376	**95**
96	0.036 3368	0.040 9485	0.045 6675	0.050 4665	0.060 2241	**96**
97	0.036 2899	0.040 9112	0.045 6383	0.050 4441	0.060 2114	**97**
98	0.036 2448	0.040 8754	0.045 6105	0.050 4227	0.060 1994	**98**
99	0.036 2012	0.040 8410	0.045 5838	0.050 4024	0.060 1880	**99**
100	0.036 1593	0.040 8080	0.045 5584	0.050 3831	0.060 1774	**100**

TABLE VIII.—COMPOUND AMOUNT FOR TIMES LESS THAN A YEAR

$$s = (1 + i)^n$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
$\frac{1}{12}$	1.001 0357	1.001 2415	1.001 6516	1.002 0598	1.002 4663	$\frac{1}{12}$
$\frac{1}{4}$	1.003 1105	1.003 7291	1.004 9629	1.006 1922	1.007 4171	$\frac{1}{4}$
$\frac{1}{2}$	1.006 2306	1.007 4721	1.009 9505	1.012 4228	1.014 8892	$\frac{1}{2}$
n	$3\frac{1}{2}\%$	4%	5%	6%	7%	n
$\frac{1}{12}$	1.002 8709	1.003 2737	1.004 0741	1.004 8676	1.005 6541	$\frac{1}{12}$
$\frac{1}{4}$	1.008 6374	1.009 8534	1.012 2722	1.014 6739	1.017 0585	$\frac{1}{4}$
$\frac{1}{2}$	1.017 3495	1.019 8039	1.024 6951	1.029 5630	1.034 4080	$\frac{1}{2}$

TABLE IX.—THE VALUE OF

$$j_{(p)} = p\left[(1 + i)^{\frac{1}{p}} - 1\right]$$

p	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	p
12	.012 4290	.014 8979	.019 8190	.024 7181	.029 5952	12
4	.012 4418	.014 9164	.019 8517	.024 7690	.029 6683	4
2	.012 4612	.014 9442	.019 9010	.024 8457	.029 7783	2
p	$3\frac{1}{2}\%$	4%	5%	6%	7%	p
12	.034 4508	.039 2849	.048 8895	.058 4106	.067 8492	12
4	.034 5498	.039 4136	.049 0889	.058 6954	.068 2341	4
2	.034 6990	.039 6078	.049 3902	.059 1260	.068 8161	2

TABLE X.—THE VALUE OF

$$\frac{i}{j_{(p)}} = \frac{i}{p\left[(1 + i)^{\frac{1}{p}} - 1\right]}$$

p	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	p
12	1.005 7124	1.006 8547	1.009 1347	1.011 4054	1.013 6760	12
4	1.004 6681	1.005 6073	1.007 4694	1.009 3262	1.011 1807	4
2	1.003 2316	1.003 7366	1.004 9743	1.006 2115	1.007 4446	2
p	$3\frac{1}{2}\%$	4%	5%	6%	7%	p
12	1.015 9420	1.018 2035	1.022 7148	1.027 2106	1.031 7024	12
4	1.013 0309	1.014 8774	1.018 5694	1.022 2268	1.025 8800	4
2	1.008 6748	1.009 9022	1.012 3475	1.014 7815	1.017 2040	2

TABLE XI.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number Living l_x	Number Dying d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x	Age x	Number Living l_x	Number Dying d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x
10	100 000	749	.007 490	.992 510	**53**	66 797	1 091	.016 333	.983 667
11	99 251	746	.007 516	.992 484	**54**	65 706	1 143	.017 396	.982 604
12	98 505	743	.007 543	.992 457	**55**	64 563	1 199	.018 571	.981 429
13	97 762	740	.007 569	.992 421	**56**	63 364	1 260	.019 885	.980 115
14	97 022	737	.007 596	.992 404	**57**	62 104	1 325	.021 335	.978 665
15	96 285	735	.007 634	.992 366	**58**	60 779	1 394	.022 936	.977 064
16	95 550	732	.007 661	.992 339	**59**	59 385	1 468	.024 720	.975 280
17	94 818	729	.007 688	.992 312	**60**	57 917	1 546	.026 693	.973 307
18	94 089	727	.007 727	.992 273	**61**	56 371	1 628	.028 880	.971 120
19	93 362	725	.007 765	.992 235	**62**	54 743	1 713	.031 292	.968 708
20	92 637	723	.007 805	.992 195	**63**	53 030	1 800	.033 943	.966 057
21	91 914	722	.007 855	.992 145	**64**	51 230	1 889	.036 873	.963 127
22	91 192	721	.007 906	.992 094	**65**	49 341	1 980	.040 129	.959 871
23	90 471	720	.007 958	.992 042	**66**	47 361	2 070	.043 707	.956 293
24	89 751	719	.008 011	.991 989	**67**	45 291	2 158	.047 647	.952 353
25	89 032	718	.008 065	.991 935	**68**	43 133	2 243	.052 002	.947 998
26	88 314	718	.008 130	.991 870	**69**	40 890	2 321	.056 762	.943 238
27	87 596	718	.008 197	.991 803	**70**	38 569	2 391	.061 993	.938 007
28	86 878	718	.008 264	.991 736	**71**	36 178	2 448	.067 665	.932 335
29	86 160	719	.008 345	.991 655	**72**	33 730	2 487	.073 733	.926 267
30	85 441	720	.008 427	.991 573	**73**	31 243	2 505	.080 178	.919 822
31	84 721	721	.008 510	.991 490	**74**	28 738	2 501	.087 028	.912 972
32	84 000	723	.008 607	.991 393	**75**	26 237	2 476	.094 371	.905 629
33	83 277	726	.008 718	.991 282	**76**	23 761	2 431	.102 311	.897 689
34	82 551	729	.008 831	.991 169	**77**	21 330	2 369	.111 064	.888 936
35	81 822	732	.008 946	.991 054	**78**	18 961	2 291	.120 827	.879 173
36	81 090	737	.009 089	.990 911	**79**	16 670	2 196	.131 734	.868 266
37	80 353	742	.009 234	.990 766	**80**	14 474	2 091	.144 466	.855 534
38	79 611	749	.009 408	.990 592	**81**	12 383	1 964	.158 605	.841 395
39	78 862	756	.009 586	.990 414	**82**	10 419	1 816	.174 297	.825 703
40	78 106	765	.009 794	.990 206	**83**	8 603	1 648	.191 561	.808 439
41	77 341	774	.010 008	.989 992	**84**	6 955	1 470	.211 359	.788 641
42	76 567	785	.010 252	.989 748	**85**	5 485	1 292	.235 552	.764 448
43	75 782	797	.010 517	.989 483	**86**	4 193	1 114	.265 681	.734 319
44	74 985	812	.010 829	.989 171	**87**	3 079	933	.303 020	.696 980
45	74 173	828	.011 163	.988 837	**88**	2 146	744	.346 692	.653 308
46	73 345	848	.011 562	.988 438	**89**	1 402	555	.395 863	.604 137
47	72 497	870	.012 000	.988 000	**90**	847	385	.454 545	.545 455
48	71 627	896	.012 509	.987 491	**91**	462	246	.532 466	.467 534
49	70 731	927	.013 106	.986 894	**92**	216	137	.634 259	.365 741
50	69 804	962	.013 781	.986 219	**93**	79	58	.734 177	.265 823
51	68 842	1 001	.014 541	.985 459	**94**	21	18	.857 143	.142 857
52	67 841	1 044	.015 389	.984 611	**95**	3	3	1.000 000	.000 000

TABLE XII.—COMMUTATION COLUMNS

AMERICAN EXPERIENCE, THREE AND ONE-HALF PER CENT

x Age	D_x	$N_x = D_x + D_{x+1} + \cdots$	M_x	x Age	D_x	$N_x = D_x + D_{x+1} + \cdots$	M_x
10	70 891.9	1 575 535.3	17 612.91	**53**	10 787.4	145 915.7	5 853.095
11	67 981.5	1 504 643.4	17 099.89	**54**	10 252.4	135 128.2	5 682.861
12	65 189.0	1 436 661.9	16 606.20	**55**	9 733.40	124 875.8	5 510.544
13	62 509.4	1 371 472.9	16 131.12	**56**	9 229.60	115 142.4	5 335.898
14	59 938.4	1 308 963.5	15 673.96	**57**	8 740.17	105 912.8	5 158.573
15	54 471.6	1 249 025.0	15 234.05	**58**	8 264.44	97 172.64	4 978.405
16	55 104.2	1 191 553.4	14 810.17	**59**	7 801.83	88 908.20	4 795.266
17	52 832.9	1 136 449.2	14 402.30	**60**	7 351.65	81 106.38	4 608.926
18	50 653.9	1 083 616.2	14 009.83	**61**	6 913.44	73 754.73	4 419.322
19	48 562.8	1 032 962.4	13 631.68	**62**	6 486.75	66 841.28	4 226.413
20	46 556.2	984 399.6	13 267.32	**63**	6 071.27	60 354.54	4 030.296
21	44 630.8	937 843.3	12 916.25	**64**	5 666.85	54 283.27	3 831.187
22	42 782.8	893 212.5	12 577.53	**65**	5 273.33	48 616.41	3 629.300
23	41 009.2	850 429.7	12 250.71	**66**	4 890.55	43 343.08	3 424.843
24	39 307.1	809 420.5	11 935.38	**67**	4 518.65	38 452.53	3 218.321
25	37 673.6	770 113.4	11 631.14	**68**	4 157.82	33 933.88	3 010.299
26	36 106.1	732 439.8	11 337.59	**69**	3 808.32	29 776.06	2 801.396
27	34 601.5	696 333.7	11 053.97	**70**	3 470.67	25 967.74	2 592.538
28	33 157.4	661 733.2	10 779.94	**71**	3 145.43	22 497.07	2 384.657
29	31 771.3	628 574.8	10 515.18	**72**	2 833.42	19 351.64	2 179.018
30	30 440.8	596 803.6	10 259.02	**73**	2 535.75	16 518.22	1 977.167
31	29 163.5	566 362.9	10 011.17	**74**	2 253.57	13 982.47	1 780.731
32	27 937.5	537 199.3	9 771.375	**75**	1 987.87	11 728.90	1 591.240
33	26 760.5	509 261.8	9 539.044	**76**	1 739.39	9 741.028	1 409.988
34	25 630.1	482 501.3	9 313.638	**77**	1 508.63	8 001.633	1 238.047
35	24 544.7	456 871.2	9 094.955	**78**	1 295.73	6 492.999	1 076.158
36	23 502.5	432 326.5	8 882.798	**79**	1 100.65	5 197.271	924.893 7
37	22 501.4	408 824.0	8 676.415	**80**	923.338	4 096.624	784.804 6
38	21 539.7	386 322.6	8 475.658	**81**	763.234	3 173.286	655.924 5
39	20 615.5	364 782.9	8 279.860	**82**	620.465	2 410.052	538.965 7
40	19 727.4	344 167.4	8 088.915	**83**	494.995	1 789.587	434.477 6
41	18 873.6	324 440.0	7 902.231	**84**	386.641	1 294.592	342.862 4
42	18 052.9	305 566.3	7 719.738	**85**	294.610	907.951 3	263.905 9
43	17 263.6	287 513.4	7 540.910	**86**	217.598	613.341 7	196.856 9
44	16 504.4	270 249.8	7 365.489	**87**	154.383	395.743 8	141.000 3
45	15 773.6	253 745.5	7 192.809	**88**	103.963	241.360 9	95.801 07
46	15 070.0	237 971.9	7 022.682	**89**	65.623 1	137.397 8	60.976 81
47	14 392.1	222 901.9	6 854.337	**90**	38.304 7	71.774 70	35.877 55
48	13 738.5	208 509.8	6 687.466	**91**	20.186 9	33.470 01	19.055 09
49	13 107.9	194 771.3	6 521.419	**92**	9.118 89	13.283 09	8.669 695
50	12 498.6	181 663.4	6 355.436	**93**	3.222 36	4.164 21	3.081 545
51	11 909.6	169 164.7	6 189.012	**94**	0.827 611	0.941 84	.795 762
52	11 339.5	157 255.2	6 021.696	**95**	0.114 232	0.114 23	.110 369

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